Algorithmic and Economic Aspects of Networks

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Beliefs in Social Networks

Given that we influence each other's beliefs,

- will we agree or remain divided?
- who has the most influence over our beliefs?
- how quickly do we learn?
- do we learn the truth?

Observational Learning

Key Idea: If your neighbor is doing better than you are, copy him.

Bayesian Updating Model

n agents connected in a social network

at each time t = 1, 2, …, each agent selects an action from a finite set

payoffs to actions are random and depend on the state of nature

Agent Goal

maximize sum of discounted payoffs

$$\sum_{t > 0} \delta^t \cdot \pi_{it}$$

where $\delta < 1$ is discount factor and π_{it} is payoff to i at time t.

Example

Two actions

action A has payoff 1

action B has payoff 2 with probability p and 0 with probability (1-p)

If $p > ½$, agents prefer B, else agents prefer A.

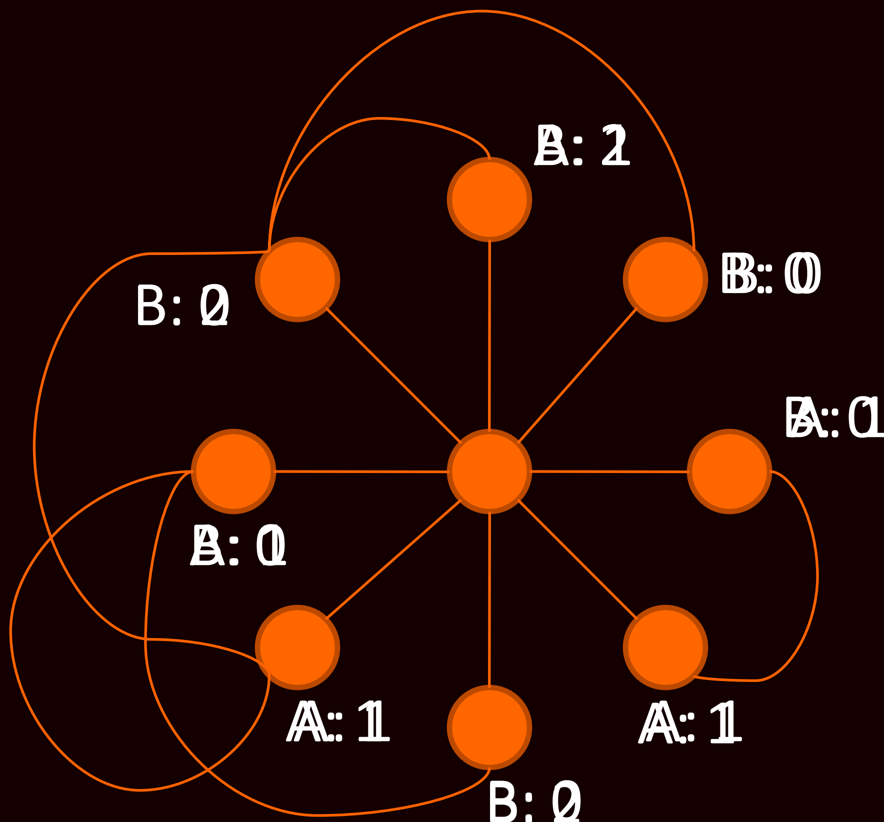
Example

Agents have beliefs $\mu_i(p_j)$ representing probability agent i assigns to event that $p = p_j$.

Multi-armed bandit

… *with observations.*

Example



Center agent, Day 0:

Pr[p=1/3] = 0, Pr[p=2/3] = 1

Play action B, payoff 0

Center agent, Day 1:

Pr[p=1/3] > 0, Pr[p=2/3] < 1

Play action A, payoff 1

Center agent, Day 2:

Now must take into account "echoes" for optimal update

Example

Ignoring echoes,

Theorem [Bala and Goyal]: With prob. 1, all agents eventually play the same action.

Proof: By strong law of large numbers, if B is played infinitely often, beliefs converge to correct probability.

Example

Note, all agents play same action, but

- don’t necessarily have same beliefs
- don’t necessarily pick “right” action *

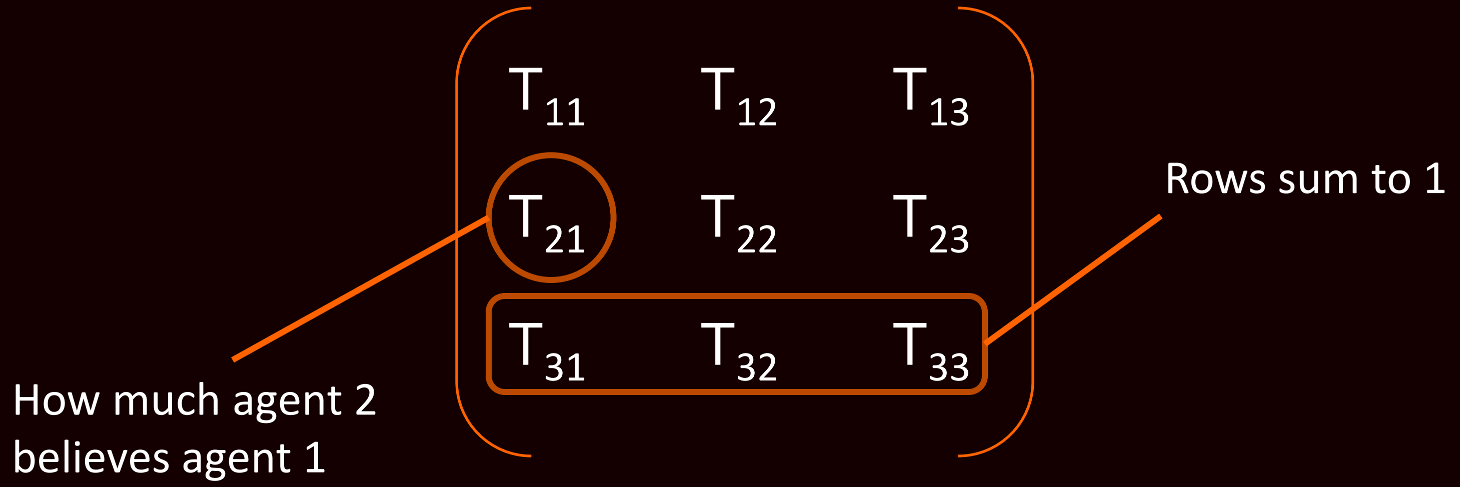
* unless someone is optimistic about B

Imitation and Social Influence

At time t, agent i has an opinion $p_i(t)$ in [0,1].

Let $p(t) = (p_1(t), \ldots, p_n(t))$ be vector of opinions.

Matrix T represents interactions:



Updating Beliefs

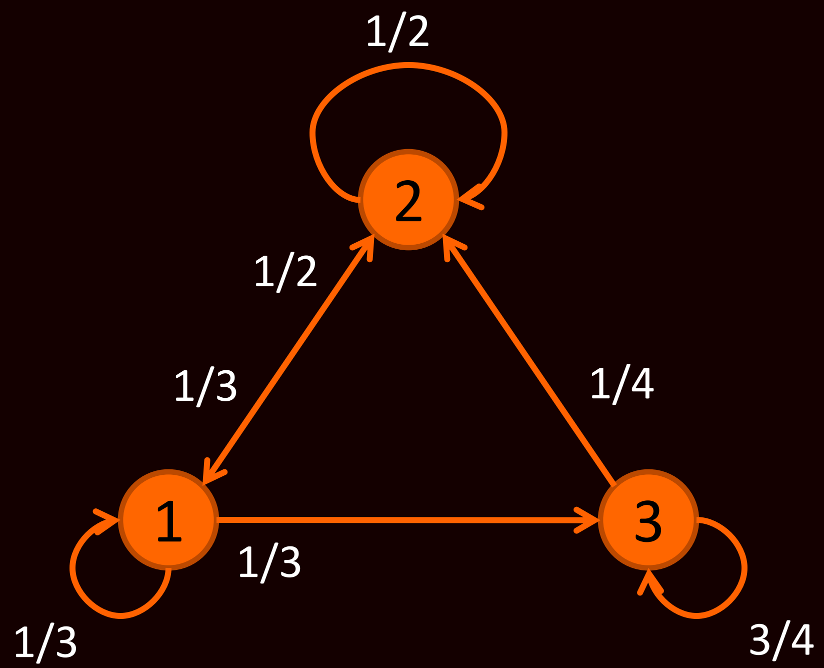
Update rule: $p(t) = T \cdot p(t-1)$

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} p_1(t-1) \\ p_2(t-1) \\ p_3(t-1) \end{pmatrix} = \begin{pmatrix} T_{11}p_1(t-1) \; T_{12}p_1(t-1) \; T_{13}p_1(t-1) \\ T_{21}p_2(t-1) \; T_{22}p_2(t-1) \; T_{23}p_2(t-1) \\ T_{31}p_3(t-1) \; T_{32}p_3(t-1) \; T_{33}p_3(t-1) \end{pmatrix}$$

Example

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$

Example



Example

Suppose p(0) = (1, 0, 0). Then

$$p(1) = T\,p(0) = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = (1/3,\ 1/2,\ 0)$$

p(2) = T p(1) = (5/18, 5/12, 1/8)

p(3) = T p(2) = (0.273, 0.347, 0.198)

p(4) = T p(3) = (0.273, 0.310, 0.235)

… p(∞) → (0.2727, 0.2727, 0.2727)

Incorporating Media

Media is listened to by (some) agents, but not influenced by anyone.

Represent media by agent i with $T_{ii} = 1$, $T_{ij} = 0$ for j not equal to i. Media influences agents k for which $T_{ki} > 0$.

Converging Beliefs

When does process have a limit?

Note $p(t) = T\, p(t-1) = T^2\, p(t-2) = \ldots = T^t\, p(0)$.

Process converges when T^t converges.

Final influence weights are rows of T^t.

Example

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{t} \longrightarrow \begin{pmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{pmatrix}$$

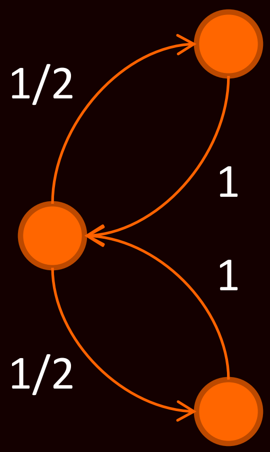
Example

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Does not converge!

Example

0	1/2	1/2
1	0	0
1	0	0



Aperiodic

Definition. T is aperiodic if the gcd of all cycle lengths is one (e.g., if T has a self loop).

Convergence

Can be relaxed, see book.

T is aperiodic and strongly connected

Everyone should trust themselves a little bit.

(standard results in Markov chain theory)

↓

T converges

Consensus

For any aperiodic matrix T, any “closed” and strongly connected group reaches consensus.

Social Influence

We look for a unit vector $s = (s_1, \ldots, s_n)$ such that

$$p(\infty) = s \cdot p(0)$$

Then s would be the relative influences of agents in society as a whole.

Social Influence

Note p(o) & T p(o) have same limiting beliefs, so

$$s \cdot p(o) = s \cdot (T\, p(o))$$

And since this holds for every p, it must be that

$$s\,T = s$$

Social Influence

The vector s is an eigenvector of T with eigenvalue one.

If T is strongly connected, aperiodic, and has rows that sum to one, then s is unique.

Another interpretation: s is the stationary distribution of the random walk.

Computing Social Influence

Since

$$s \cdot p(0) = p(\infty) = T^{\infty} \cdot p(0)$$

it must be that each row of T converges to s.

Who's Influential?

Note, since s is an eigenvector, $s_i = \sum T_{ji}\, s_j$, so an agent has high influence if they are listened to by influential people.

PageRank

Compute influence vector on web graph and return pages in decreasing order of influence.

- each page seeks advice from all outgoing links (equally)

- add restart probabilities to make strongly connected

- add initial distribution to bias walk

Time to Convergence

If it takes forever for beliefs to converge, then we may never observe the final state.

Time to Convergence

Two agents

1. similar weightings ($T_{11} \sim T_{21}$) implies fast convergence

2. different weightings ($T_{11} >> T_{21}$) implies slow convergence

Diagonal Decomposition

Want to explore how far T^t is from T^∞

Rewrite T in its diagonal decomposition so

$$T = u^{-1} \Lambda u$$

for a matrix u and a *diagonal matrix* Λ.

1. Compute eigenvectors of T
2. Let u be matrix of eigenvectors
3. Let Λ be matrix of eigenvalues

Exponentiation

Now T^t becomes:

$$(u^{-1} \Lambda u) (u^{-1} \Lambda u) \ldots (u^{-1} \Lambda u)$$

$$=$$

$$u^{-1} \Lambda^t u$$

and Λ^t is diagonal matrix, so easy exponentiate.

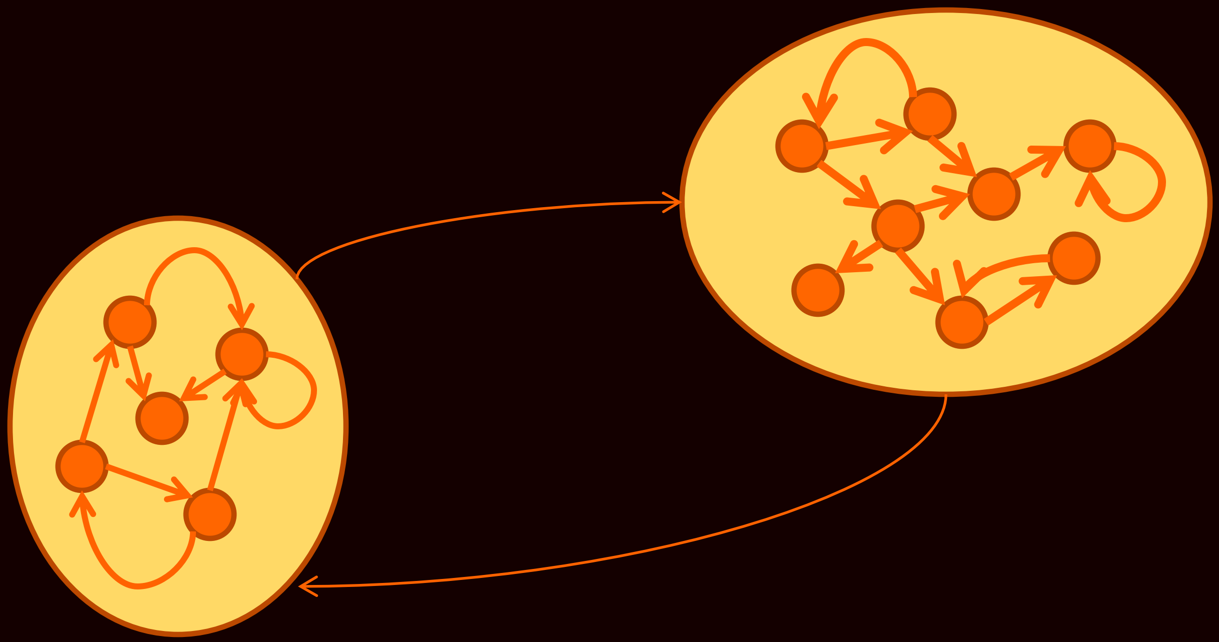
Speed of Convergence

$$\begin{pmatrix} 1 & 0 \\ 0 & T_{11} - T_{12} \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ 0 & (T_{11} - T_{12})^t \end{pmatrix}$$

Since $(T_{11} - T_{12}) < 1$, $(T_{11} - T12)^t$ converges to zero.
Speed of convergence is related to magnitute of 2nd eigenvalue,
… and to how different weights are.

More Agents

Speed of convergence now relates to how much groups trust each other.



Finding the Truth

When do we converge to the correct belief?

Assume Truth Exists

There is a ground truth μ.

There are n agents (to make formal, study sequence of societies with $n \rightarrow \infty$).

Each agent has a signal $p_i(o)$ distributed with mean μ and variance σ_i^2.

Wisdom

Definition. Networks are **wise** if $p(\infty)$ converges to μ when n is large enough.

Truth Can Be Found

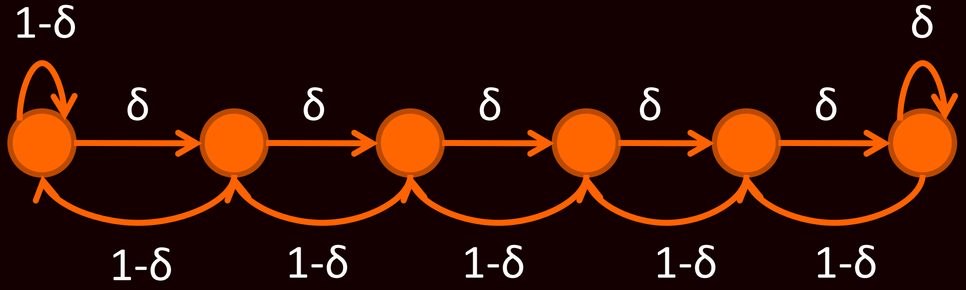
By law of large numbers, averaging all beliefs with equal weights converges to truth.

Sufficient: agents have equal influence.

Necessary Conditions

Necessary that

- no agent has too much influence
- no agent has too much relative influence
- no agent has too much indirect influence



Sufficient Conditions

Sufficient that the society exhibits

- balance: a smaller group of agents does not get infinitely more weight in from a larger group than it gives back
- dispersion: each small group must give some minimum amount of weight to larger groups

Assignment:

- Readings:
  - Social and Economic Networks, Chapter 8
  - PageRank papers
- Reaction to paper
- Presentation volunteer?