# Algorithmic and Economic Aspects of Networks

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# Network Formation

How do we pick our friends?

# Picking Friends

Based on …

chance?

relatives, teachers, roommates

or more of a quid-pro-quo?

professional societies, study groups, your SO

# Friends with Benefits

Having friends incurs a cost …

and also offers a benefit.

$u_i(G)$ = net benefit to i of social network G

# Friends with Benefits

The more distant a friend, the less the benefit.

Let b map distance to benefit:

b(d(ij)) = benefit to i of j at distance d(ij)

Then utility to i in network G is:

$$u_i(G) = \sum_j b(d(ij)) - c \cdot \deg(i)$$

Cost of link formation.

# Life is a Game

Players: $V = \{1, \ldots, n\}$

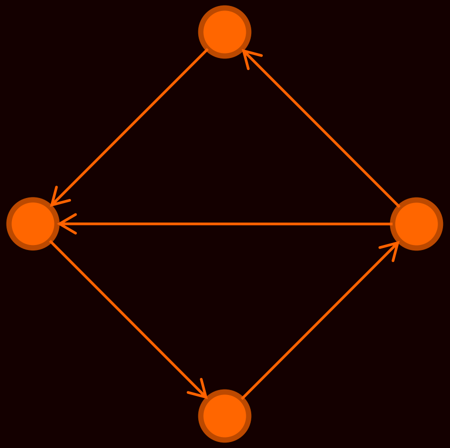
Strategies: S in $\{1, \ldots, n\}$

Outcome is (directed network) $G(V,E)$

where $(ij)$ in $E$ if $j$ in $S_i$

# Equilibria

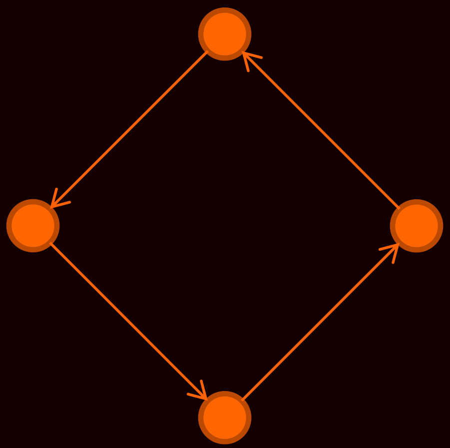
No player unilaterally wants to change strategy.



$u_i(G)$ = # nodes i can reach - # of links formed

# Strict Equilibria

Any change *strictly decreases* some player's utility.



$u_i(G)$ = # nodes i can reach - # of links formed

# Information Flows

One-way flow: A link can be used *only* by the person who formed it to send information

Two-way flow: A link between two people can be used by *either person*

# Equilibrium Networks

Bala and Goyal, 2000:

- Every equilibrium is connected or empty
- For one-way flow, only strict equilibria are the directed cycle and/or empty network
- For two-way flow, only strict equilibria are center-sponsored star and/or empty network

# Equilibrium Selection

Best-response dynamics:

- Start from an arbitrary initial graph
- In each period, each player independently decides to “move” with probability p
- If a player decides to move, he picks a new strategy randomly from his set of best responses to graph in previous period

# Equilibrium Selection

Theorem: In either model, the dynamic process converges to a strict equilibrium network with probability one.

… rapidly, according to simulations

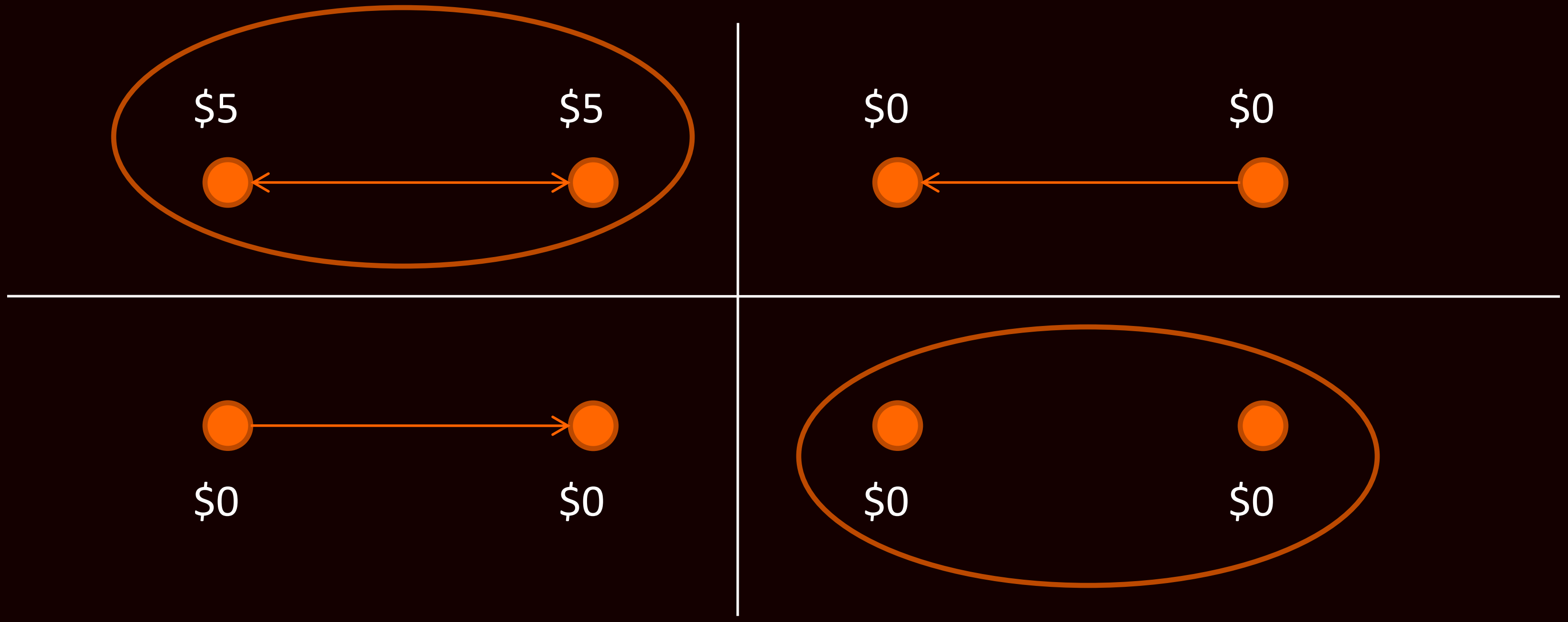
# Modeling Consent

A relationship is a two-way street.

It takes two to make it,
and one to break it.

# Modeling Consent

Players each earn $5 if form relationship.



# Pairwise Stability

Definition. A network G is pairwise stable if

1. No player wants to sever existing link ij:

$$u_i(G) \geq u_i(G - ij)$$

2. No pair wants to form non-existing link ij:

$$\text{If } u_i(G + ij) > u_i(G), \text{ then } u_j(G + ij) < u_j(G)$$

# Pairwise Stable Networks

Recall $u_i(G) = \sum_j b(d(ij)) - c \cdot deg(i)$.

Observation: A pairwise stable network has at most one non-empty component.

Proof: For any link to form, must have $c < b(1)$, so all nodes will be connected.

# Pairwise Stable Networks

1. If forming links is cheap ($b(2) < b(1) - c$), only pairwise stable network is complete one.

2. If forming links is expensive ($b(1) < c$), only pairwise stable network is empty one.

3. For intermediate costs ($b(1) - b(2) < c < b(1)$), stars are pairwise stable.

# Efficiency

A network G is efficient if

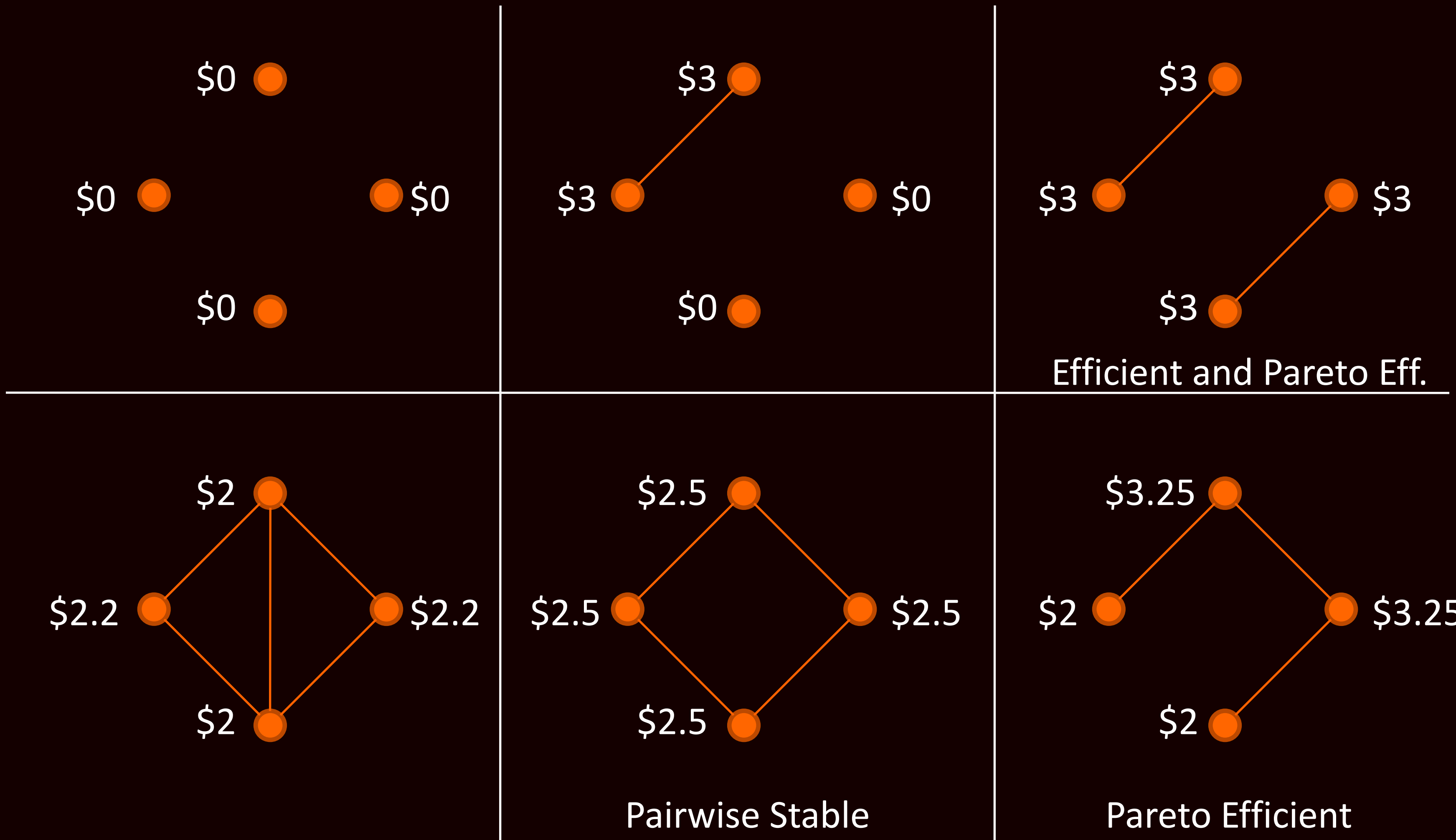
$\sum_i u_i(G) > \sum_i u_i(G')$ for all networks G'.

# Pareto Efficiency

Network G is pareto efficient if there is no G' s.t.

$u_i(G) \geq u_i(G')$ for all i and strict for some i.

# Efficiency vs Pareto Efficiency



# Efficient Networks

Recall $u_i(G) = \sum_j b(d(ij)) - c \cdot deg(i)$.

Thm. The unique efficient network structure is

1. the complete network if $b(2) < b(1) - c$,
2. a star encompassing all nodes if $b(1) - b(2) < c < b(1) + (n-2)b(2)/2$, and
3. the empty network if $b(1) + (n-2)b(2)/2 < c$.

# Efficiency of Equilibria

For high and low costs, all equilibria are efficient.

For intermediate costs, equilibria may not be efficient.

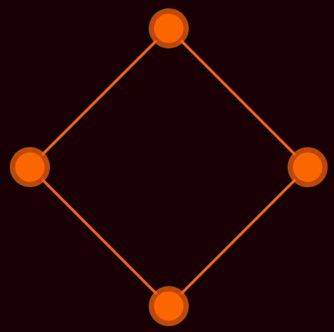
# The Virtue of Selfishness

Can we quantify how much is lost due to selfish behavior of agents?

Definition. The price of anarchy is the ratio of the worst equilibrium cost to the socially optimal cost.

# Example

Fabrikant et al., 2003: $u_i(G) = \sum_j -d(ij) - c \cdot deg(i)$.

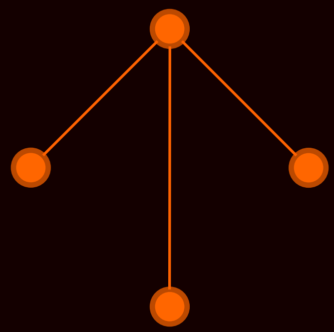


Social cost = 4 x (2c + 4) = 8c + 16

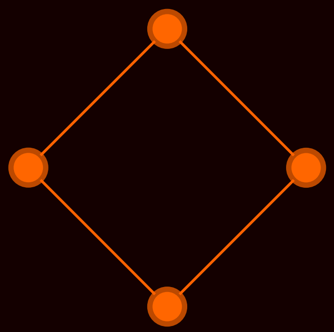
# Example

Fabrikant et al., 2003: $u_i(G) = \sum_j -d(ij) - c \cdot deg(i)$.

Suppose c = 2. Price of anarchy is ≥ 16/15.



Socially optimal network
cost = 9 + 3 x 7 = 30



A stable network
cost = 8 x 2 + 16 = 32

# Example

Recall $u_i(G) = \sum_j -d(ij) - c \cdot deg(i)$.

1. What are the efficient networks?

   c < 1 → the complete graph

   c > 1 → a star

2. What are the stable networks?

   c < 1 → the complete graph

   c > 1 → a star …

# Example

Fabrikant et al., 2003

Let $u_i(G) = \sum_j -d(ij) - c \cdot deg(i)$.

Thm. The price of anarchy is at most $(17 \cdot \sqrt{c})$.

Proof Sketch. On board.

# Externalities

Our actions impact those around us.

Positive impact = positive externalities

Negative impact = negative externalites

# Externalities

Positive externalities

Fabrikant et al.: $u_i(G) = \sum_j -d(ij) - c \cdot deg(i)$.

Negative externalities

Jackson and Wolinsky: co-authorship model.

# Co-authorship

$$u_i(G) = \sum_j 1/deg(j) + 1/deg(i) + 1/(deg(j).deg(i))$$

Amount of time i spends on project

Amount of time j spends on project

Amount of time i spends working with j on project

# Co-authorship

**Theorem**. If n is even and $n > 3$, then

1. the efficient network consists of $n/2$ separate pairs
2. pairwise stable networks are inefficient and consistent of components of geometrically growing size.

**Proof**. In book.

# Inefficiency

In both models, inefficiencies arise because of externalities. That is, individuals do not account for global effect of local actions.

Fixes: taxes, subsidies, …

# Assignment:

- Readings:
  - Social and Economic Networks, Chapter 6 (Chapter 11 optional)
  - J. Kleinberg, S. Suri, E. Tardos, and T. Wexler. *Strategic Network Formation with Structural Holes*. ACM Conference on Electronic Commerce, 2008.
- Reaction to Kleinberg et al, or paper of your choice
- Project proposals due 12/2/2009.
- Presentation volunteer? Arun.