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(a)

$$\min_{x\in\Delta_m}\max_{y\in\Delta_n} y^T AX = \min_{x\in\Delta_m}\max_{y\in\Delta_n}\begin{pmatrix} y_1 & y_2 & ... & y_n \end{pmatrix}\begin{pmatrix} \sum_{j=1}^m a_{1j}x_j \\ \sum_{j=1}^m a_{2j}x_j \\ . \\ . \\ . \\ \sum_{j=1}^m a_{nj}x_j \end{pmatrix} \tag{1}$$

For a fixed x, we assume $\max_i \sum_{j=1}^m a_{ij}x_j = \sum_{j=1}^m a_{kj}x_j$.

$$\therefore y^T AX \le \max_i \sum_{j=1}^m a_{ij}x_j(y_1+y_2+...+y_n) = \sum_{j=1}^m a_{kj}x_j \tag{2}$$

At the same time, when $y_k = 1$, $y^T AX = \sum_{j=1}^m a_{kj}x_j$ $\therefore y_k = 1$ can make $y^T AX$ maximum, which means for y there is a best response is a pure strategy.

Similarly, for $\max_{y\in\Delta_n}\min_{x\in\Delta_m} y^T AX$, we can find a pure strategy for x.

$$\therefore \max_{y\in\Delta_n}\min_{j\in\{1..m\}} y^T AX = \min_{x\in\Delta_m}\max_{i\in\{1..n\}} y^T AX \tag{3}$$

(b)

First, we will give a solution for a simplified version. We have the following assumptions. A is square. $\det a \neq 0$. And we only target at mixed strategy.

Let's look back at the equation.

Left Side $= \min_{x\in\Delta_m}\max\left(\sum_{j=1}^m a_{1j}x_j, \sum_{j=1}^m a_{2j}x_j, ..., \sum_{j=1}^m a_{nj}x_j\right)$

The space is a combination of n linear functions. For each linear function, the min point can only be at its edge.

So min point is either $0, 0, ...1, ..0, 0$ or intersection of linear functions. We skip pure strategy. Intersection of any two linear function is still a lower dimension space. So its min point will still be the edge.

So the min point will be the intersection of all linear functions.

$$\therefore \begin{cases} x_1 + x_2 + ... + x_n = 1 \\ \sum_{j=1}^m a_{1j}x_j = \sum_{j=1}^m a_{2j}x_j \\ \vdots \\ \sum_{j=1}^m a_{(n-1)j}x_j = \sum_{j=1}^m a_{nj}x_j \end{cases} \tag{4}$$

$$\therefore \begin{pmatrix} 1 & 1 & ... & 1 \\ a_{11}-a_{21} & a_{12}-a_{22} & ...1 & a_{1n}-a_{2n} \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{(n-1)1}-a_{n1} & a_{(n-1)2}-a_{n2} & ... & a_{(n-1)n}-a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \tag{5}$$

$$\therefore x_j = \frac{\det\begin{pmatrix} 1 & 1 & ... & 1 & ... & 1 \\ a_{11}-a_{21} & a_{12}-a_{22} & ... & 0 & ... & a_{1n}-a_{2n} \\ . & & & 0 & & . \\ . & & & 0 & & . \\ . & & & 0 & & . \\ a_{(n-1)1}-a_{n1} & a_{(n-1)2}-a_{n2} & ... & 0 & ... & a_{(n-1)n}-a_{nn} \end{pmatrix}}{\;}$$

$$= \frac{a_{nj}(-1)^j \det \begin{pmatrix} a_{n1} & a_{n2} & ... & a_{nn} \\ a_{11}-a_{21} & a_{12}-a_{22} & ... & a_{1n}-a_{2n} \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{(n-1)1}-a_{n1} & a_{(n-1)2}-a_{n2} & ... & a_{(n-1)n}-a_{nn} \end{pmatrix}}{(-1)^n \sum_{i,j=1}^{n} A_{ij}} = \frac{\det A}{(-1)^n \sum_{i,j=1}^{n} A_{ij}} \tag{8}$$

Similarly, Right=$\frac{\det A}{(-1)^n \sum_{i,j=1}^{n} A_{ij}}$

$\therefore left = right$

For a general case, there are many points of intersections. We need to consider all of them. It is not hare. So we go back to the classical prove.

let $Z = \max\left(\sum_{j=1}^{m} a_{1j}x_j, \sum_{j=1}^{m} a_{2j}x_j, ..., \sum_{j=1}^{m} a_{nj}x_j\right)$

We have LP programming problem.

$$\begin{cases} Ax \le Z \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\ \sum x_j = 1 \\ \forall j x_j \ge 0 \end{cases} \tag{9}$$

$$\min\,(x_1, ..., x_m) Z \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = Z \tag{10}$$

Its dual problem is

$$\begin{cases} A^T y \ge Z \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\ \sum y_i = 1 \\ \forall i y_i \ge 0 \end{cases} \tag{11}$$

$$\max\,(y_1, ..., x_n) Z \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = Z \tag{12}$$

This is equal to $\max_{y \in \Delta_n} \min_{j \in \{1..m\}} y^T AX$.

(c)

Deterministic Algorithm: We always choose the element with the smallest index in the sets as a pivot to divide sets into two.

Input: uniform distribution

First, we try to prove elements in each set still obey uniform distribution.

Assume x_k means the pivot is in the kst positoin.

$$P(xisist|x < x_k) = \frac{P(x < x_k|xisist)P(xisist)}{P(x < x_k)} = \frac{1/nP(x < x_k|xisist)}{(k-1)/n} \quad (13)$$

if $i < k$ $P = 1/(k-1)$ else P=0

This proves that elements in each set still obey uniform distribution.

Let T(n) represent expectation of time to cosume.

$$T(n) = (n-1) + 1/n \sum_{i=0}^{(n-1)} (T(i) + T(n-1-i)) \quad (14)$$

let $S(n) = \sum_i T(i)$
So $T(n) = (n-1) + 2/n * S(n-1)$
$S(n) = T(n) + S(n-1)$
$\therefore S(n) = S(n-1) + 2/(n-1) * S(n-2) + (n-2)$
S(1) obeys $S(1) = O((n+1)^2 \log (n+1))$
Assume S(k) obeys it, which means $S(k) < c(k+1)^2 \log (k+1)$
$S(n) < cn^2 logn + 2c(n-1)log(n-1) + (n-2) < c(n+1)^2 \log (n+1)$
So $S(n) = O(n^2 logn)$
$\therefore T(n) = O(nlogn)$
This means that quick sort is optimal.