EECS 495 Homework1: Max-flow Min-Cut Through LP Duality

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Let $G = (V, E)$ be the graph with source s and sink t, and positive edge/arc capacities c.

1 (a)

By the hint, a fictitious arc is introduced from t to s, so that flow conservation constraint holds for every vertex in V. Then the max $s - t$ flow problem can be formulated as the following LP. Let $f_{ij} \geq 0$ be the flow from vertex i to vertex j for all arc (i, j)

$$
\begin{aligned}
(LP_p) \qquad & \max_{\{f_{ij}\}} \quad f_{ts} & & \\
& \text{s.t.} \quad f_{ij} \leq c_{ij} & \forall (i,j) \in E \qquad & (x_{ij}) \\
& \sum_{\{j:(i,j)\in E\}} f_{ji} - \sum_{\{j':(j',i)\in E\}} f_{ij'} = 0 & \forall i \in V \qquad & (y_i) \\
& f_{ij} \geq 0 & \forall (i,j) \in E &
\end{aligned}
$$

2 (b)

The dual of the above problem can be written as

$$
\begin{aligned}
(LP_d) \qquad & \min_{\{x_{ij}\},\{y_i\}} \quad \sum_{(i,j)\in E} c_{ij} x_{ij} & \\
& \text{s.t.} \quad -y_i + y_j + x_{ij} \geq 0 & \forall (i,j) \in E \\
& y_s - y_t = 1 & \\
& x_{ij} \geq 0 & \forall (i,j) \in E \\
& y_i \ \textit{unrestriced} &
\end{aligned}
$$

3 (c)

The integral version LP_d is

$$
\begin{aligned}
(IP_d) \qquad & \min_{\{x_{ij}\},\{y_i\}} \quad \sum_{(i,j)\in E} c_{ij} x_{ij} & \\
& \text{s.t.} \quad -y_i + y_j + x_{ij} \geq 0 & \forall (i,j) \in E \\
& y_s - y_t = 1 & \\
& x_{ij} \in \{0, 1\} & \forall (i,j) \in E \\
& y_i \in \{0, 1\} & \forall i \in V
\end{aligned}
$$

IP_d has an interpretation as a min $s-t$ problem. Let (X_1, X_2) be a $s-t$ cut of G where $s \in X_1$ and $t \in X_2$. Let $y_i = 1$ if vertex $i \in X_1$; $y_i = 0$ if $i \in X_2$. (It is easy to see that $y_s = 1$ and y_t=0.) Then $x_{ij} = 1$ only when $(i,j) \in X_1 \times X_2$. Therefore, the objective in IP_d is exactly the capacities of a $s-t$ cut. The solution to this problem must be a min $s-t$ cut.

Since it is easy to see that both LP_p and LP_d are feasible and bounded, we have $LP_p = LP_d$ by *Strong Duality*. On the other hand, we have $IP_d \geq LP_d$ due to *integrality gap*. Therefore, we have $IP_d \geq LP_d = LP_p$, i.e. the max $s-t$ flow lower bounds the min $s-t$ cut.

4 (d)

Without loss of generalities, we add the constraint of $y_i \geq 0$ to LP_d. This is okay since it is only the difference between y_i and y_j matters in LP_d. Let the resulting problem be LP'_d.

$$
\begin{aligned}
(LP'_d) \qquad & \min_{\{x_{ij}\},\{y_i\}} && \sum_{(i,j)\in E} c_{ij} x_{ij} && \\
& \text{s.t.} && -y_i + y_j + x_{ij} \geq 0 && \forall (i,j) \in E \\
& && y_s - y_t = 1 && \\
& && x_{ij} \geq 0 && \forall (i,j) \in E \\
& && y_i \geq 0 && \forall i \in V
\end{aligned}
$$

We need to show that LP'_d is integral, i.e. the polytope is integral. We will show this by showing that its constraint matrix is *totally unimodular* (TUM).

The constraint matrix of LP'_d can be written in the form of $[X, Y]$, where X corresponds to columns of x_{ij} variables and Y corresponds to columns of y_i variables. It is easy to see that X is an identity matrix and Y is the incidence matrix of a directed graph, which is know to be totally unimodular. Furthermore, the concatenation of a TUM matrix and an identity matrix is TUM. Therefore, the constraint matrix of LP'_d is TUM. Thus, LP'_d is integral, and LP_d is also integral. This concludes that max s-t flow equals the min s-t cut .