EECS 495: Randomized Algorithms	Lecture 12
Online Primal-Dual	

Reading: Text: Vazirani, Chapter 21

[[*Recall sparest-cut, argued good cut-packings for l_1 metric solutions to LP would give good approximations and showed how to pack cuts into l_1. Now need to show how to embed LP metric into l_1.*]]

Embedding into l_1

Idea: Use distances to sets:

- Pick $S_1, \ldots, S_m$
- Define $\sigma_i(u) = \min_{t \in S_i} d(u,t)/m$
- No stretch since for dim. i and edge (u,v),

$$d(u,t_u) \leq d(u,t_v) \leq d(v,t_v) + d(u,v).$$

Question: How to choose S_i for to not over-shrink?

Idea: Choose randomly!

Algorithm: For $1 \leq i \leq \log n$, let S_i include each $t \in V$ with prob. $1/2^i$.

Claim: $O(\log n)$-distortion embedding.

Let $B(t,r)$ be ball of radius r around t. Bound $d(u,v)$:

- Let $r_1 \geq r_2 \geq 0$
- Suppose $S_i \cap B(u,r_1) = \emptyset$ and $(S_i \cap B(v,r_2) \neq \emptyset)$
- Then $|\sigma_i(u) - \sigma_i(v)| \geq (r_1 - r_2)/l$

Claim: Let A and B be disjoint subsets s.t. $|A| < 2^i$ and $|B| \geq 2^{i-1}$. Then $\Pr[(S_i \cap A = \emptyset) \wedge (S_i \cap B \neq \emptyset)] \geq c$ for some constant c.

Proof: Sets disjoint so independent, easy to calculate.

Let ρ_i be min radius s.t. both $B(u,\rho_i)$ and $B(v,\rho_i)$ have at least 2^i points.

- Suppose ρ_i limited by u
- Suppose $\rho_i < d(u,v)/2$
- $A = B^o(u,\rho_i)$, $B = B(v,\rho_{i-1})$
- Then S_i contributes $c(\rho_i - \rho_{i-1})/l$ to i'th coordinate in expectation
- Summing over coordinates up to $\rho_i = d(u,v)/2$ gives result for (u,v)

Get for all edges whp by running $\log n$ times and using Chernoff, hence dimension of embedding is $O(\log^2 n)$.

Online Primal-Dual

Problem: Ski Rental. Stay at resort for K days, K unknown

- renting costs \$1
- buying costs \$$B$

Goal: min. cost w.r.t. offline opt.

Question: Competitive soln?

- rent B days, then buy – 2-approx
- online primal-dual – $(\frac{e}{e-1})$-approx

LP formulation

Variables:

- z_j indicates if we rent skis on day j
- x indicates if we buy skis

Primal:

$$\begin{array}{ll} \min & Bx + \sum_{j=1}^{K} z_j \\ s.t. & x + z_j \geq 1, \ \forall\, j \end{array}$$

Dual:

$$\begin{array}{ll} \max & \sum_j y_j \\ s.t. & \sum_{j=1}^{K} y_j \leq B \\ & y_j \leq 1 \end{array}$$

Complementary Slackness

Claim: Given

- primal program $\min c \cdot x \; s.t. \; Ax \geq b$
- corresponding dual program $\max b \cdot y \; s.t. \; A^T y \leq c$

and primal soln x, dual soln y, s.t.

- primal slackness: for $\alpha \geq 1$ if $x_i > 0$, then $c_i/\alpha \leq \sum_j a_{ij} y_i \leq c_i$.
- dual slackness: for $\beta \geq 1$ if $y_j > 0$, then $b_j \leq \sum_i a_{ij} x_i \leq b_j \beta$

then

$$\sum_i c_i x_i \leq \alpha\beta \sum_j b_j y_j.$$

[[*And hence these solns are within an $\alpha\beta$ approx of opt.*]]

Proof:

$$\begin{aligned} \sum_i c_i x_i &\leq \sum_i \left(\alpha \sum_j a_{ij} y_j \right) x_i \\ &= \alpha \sum_j \left(\sum_i a_{ij} x_i \right) y_j \\ &\leq \alpha \sum_j (\beta b_j)\, y_j \\ &= \alpha\beta \sum_j b_j y_j \end{aligned}$$

Online LP Solving

Note: Online, so

- primal constraints/dual variables appear sequentially
- x can only increase

Idea: Solve fractional primal/dual online and round, bounding expected approx. with dual.

Example: On day j, if $x = 0$ (i.e., j'th primal constraint not satisfied):

- increase y_j until some constraint goes tight
- set corresponding primal variable to 1

[[*Same as simple alg.*]]

Analysis: (via primal-dual)

- $y_j > 0$ means $1 \leq x + z_j \leq 2$
- $x > 0$ means $\sum_j y_j = B$
- $z_j > 0$ means $y_j = 1$

so 2-approx by complementary slackness.

Algorithm: Initialize $x = 0$. On day j, if $x < 1$:

1. $z_j \leftarrow 1 - x$
2. $x \leftarrow x(1 + 1/B) + 1/(cB)$ (c determined later)
3. $y_j \leftarrow 1$

Analysis:

- primal/dual soln feasible
- ratio between change in primal/dual bounded by $(1 + 1/c)$

Claim: Feasibility.

Proof: Just need to show $\sum_j y_j \leq B$, i.e., $x \geq 1$ after B days:

- $x_j = (1 + 1/B)x_{j-1} + (1/cB)$, so $x_B = \frac{1}{cB}\sum_{j=0}^{B}(1 + 1/B)^j = \frac{(1+1/B)^B - 1}{c}$
- Picking $c = (1+1/B)^B - 1 \approx e - 1$ makes dual feasible

Claim: Gap bounded.

Proof: Compare primal and dual change in steps where they increase.

- ΔDual is 1
- ΔPrimal $= B\Delta x + z_j = B \times \frac{1}{B}\left(x + \frac{1}{c}\right) + 1 - x = \frac{1}{c} + 1 \approx \frac{e}{e-1}$

Claim: $\frac{e}{e-1}$ approx.

Proof: Total primal at most $\frac{e}{e-1}$ times total dual, follows by weak duality.

[[*Technique called primal-dual – a combinatorial algorithm simultaneously updates primal/dual variables where primal vars capture cost. Keep gap bounded and thereby prove approx. factor.*]]

Online Rounding

Algorithm: Choose $\alpha \in_R [0, 1]$. Whenever primal variable crosses α, round to 1.

Analysis: Prob. x (resp. z_j) set to 1 equals x (resp. z_j), so result follows by linearity of expectation.

[[*Note new rounding technique here – variables rounded dependently. Required for monotonicity (which is required for online constraint) and for primal feasibility.*]]

Basic approach

Given covering/packing problem:

Primal:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} c_i x_i \\ s.t. \quad & \\ \forall\, 1 \leq j \leq m \quad & \sum_{i=1}^{n} a_{ij} x_i \geq b_j \\ \forall\, 1 \leq i \leq n \quad & x_i \geq 0 \end{aligned}$$

Dual:

$$\max \quad \sum_{j=1}^{m} b_j y_j$$

$$\begin{aligned} s.t. & \\ \forall\, 1 \le i \le n \quad & \sum_{j=1}^{m} a_{ij} y_j \le c_i \\ \forall\, 1 \le j \le m \quad & y_j \ge 0 \end{aligned}$$

Note: All coefficients non-negative.

Consider special case with 0/1 coeff., so that each covering constraint j has set $S(j)$ of elts. i included in it:

Primal:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} c_i x_i \\ s.t. & \\ \forall\, 1 \le j \le m \quad & \sum_{i \in S(j)}^{n} x_i \ge 1 \\ \forall\, 1 \le i \le n \quad & x_i \ge 0 \end{aligned}$$

Dual:

$$\begin{aligned} \max \quad & \sum_{j=1}^{m} y_j \\ s.t. & \\ \forall\, 1 \le i \le n \quad & \sum_{j : i \in S(j)} y_j \le c_i \\ \forall\, 1 \le j \le m \quad & y_j \ge 0 \end{aligned}$$

Problem: Online covering:

- know: cost function c_i
- online: constraints $S(j)$
- soln: only increase variables x_i

Example: online set cover

Problem: Online packing:

- known: values
- online: packing constraints
- soln: only set y_j in round j

Example: online matching, MSVV

Three algorithms: **Algorithm:** While $\sum_{i \in S(j)} x_i < 1$:

1. for each $i \in S(j) : x_i \leftarrow x_i(1 + 1/c_i) + 1/(|S(j)|c_i)$
2. $y_j \leftarrow y_j + 1$

Let $d = \max_j |S(j)| \le m$.

Claim: Algs produce fractional covering/packing solns that're $O(\log d)$-competitive.

Proof: (of 1):

- Alg produces feasible covering soln.: obvious
- In each iter., $\Delta P \le 2\Delta D$: $\Delta D \le 1$ and for primal,

  $$\Delta P = \sum_i c_i \Delta x_i = \sum_i c_i \left(\frac{x_i}{c_i} + \frac{1}{|S(j)|c_i}\right) \le 2$$

  since $x_i < 1$ at time of update.
- Packing constraints violated by at most $O(\log d)$ (so can scale dual updates by $O(\log d)$ to get feasible packing – dual fitting)