Market Design: Lecture 4

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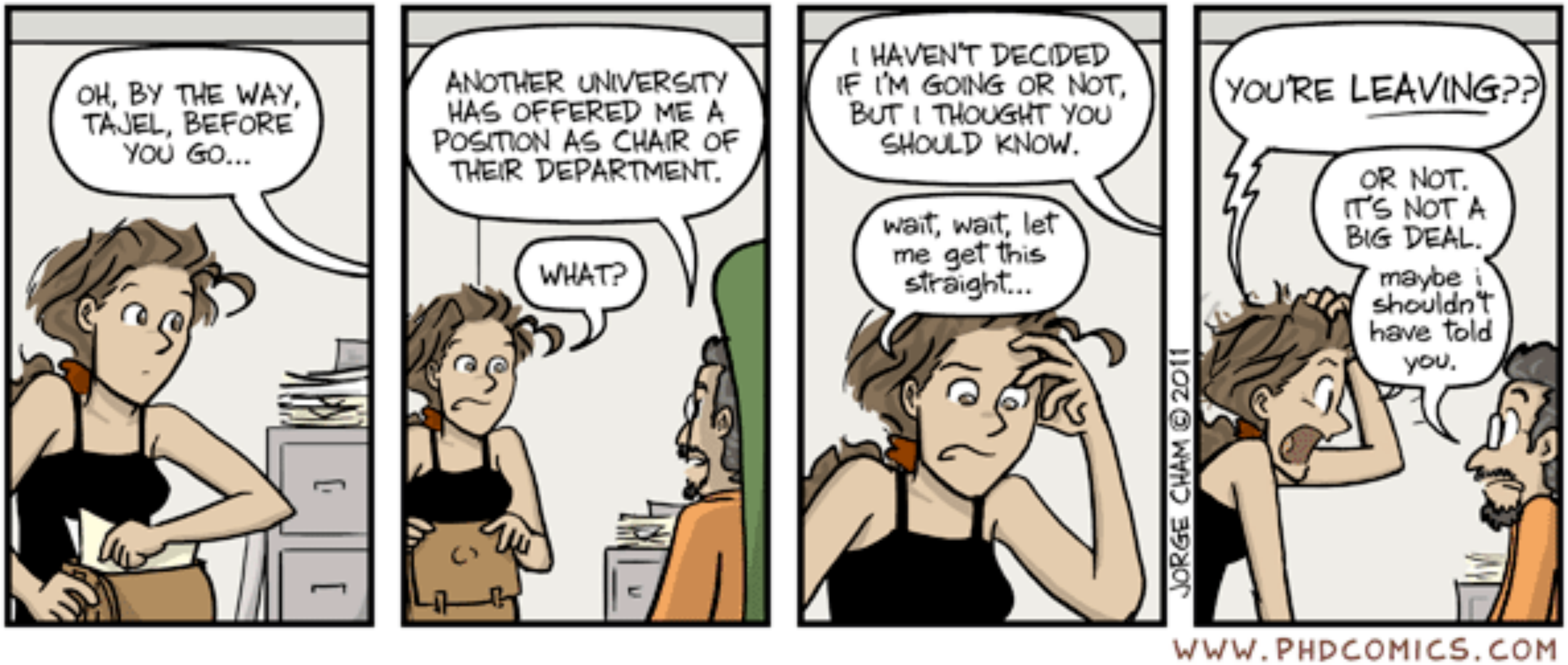
OH, BY THE WAY, TAJEL, BEFORE YOU GO...
ANOTHER UNIVERSITY HAS OFFERED ME A POSITION AS CHAIR OF THEIR DEPARTMENT.
WHAT?
I HAVEN'T DECIDED IF I'M GOING OR NOT, BUT I THOUGHT YOU SHOULD KNOW.
wait, wait, let me get this straight...
JORGE CHAM © 2011
YOU'RE LEAVING??
OR NOT. IT'S NOT A BIG DEAL.
maybe i shouldn't have told you.
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Recap

4. b) Incentives: complete information Nash equilibria, incomplete information
5. a) Many-to-one markets: responsive preferences

Outline

5. b) Many-to-one markets: substitutable preferences
6. Large market results: incentives, couples

Part 5: Many-to-one Markets.

Matching with Contracts

- doctors $D = \{d_1, \ldots, d_n\}$
- hospitals $H = \{h_1, \ldots, h_p\}$
- contracts $X = \{x\}$, each containing one doctor x_D and one hospital x_H
  - college admissions: $X = D \times H$
  - worker-firm (Kelso-Crawford): $X = D \times H \times W$ for discrete set of wages W

Matching with Contracts

- doctor d can sign at most one contract, pref. P(d) given by total order on $\{x : x_D = d\}$
- hospital h has preferences over sets of contracts, each doctor appears at most once

Choice Sets

- Func. $C_a(X')$ outputs subset of X' that a prefers
- For any subset X' of X,
  - $C_d(X')$ in {x' in X' : x'_D = d} U {null}
  - $C_d(X') >_d x$ for all other x in X' U {null}
- For any subset X' of X,
  - $C_h(X') \subseteq$ {x' in X' : x'_H = h} U {null}
  - $\{x, y\} \subseteq C_h(X')$ implies $x_D \neq y_D$
  - $C_h(X') >_h S$ for any other subset S in X' U {null}

Notation

- chosen set for doctors $C_D(X') = \bigcup_d C_d(X')$
- chosen set for hospitals $C_H(X') = \bigcup_h C_h(X')$
- rejection set for doctors $R_D(X') = X' - C_D(X')$
- rejection set for hospitals $R_H(X') = X' - C_H(X')$

Stable Allocations

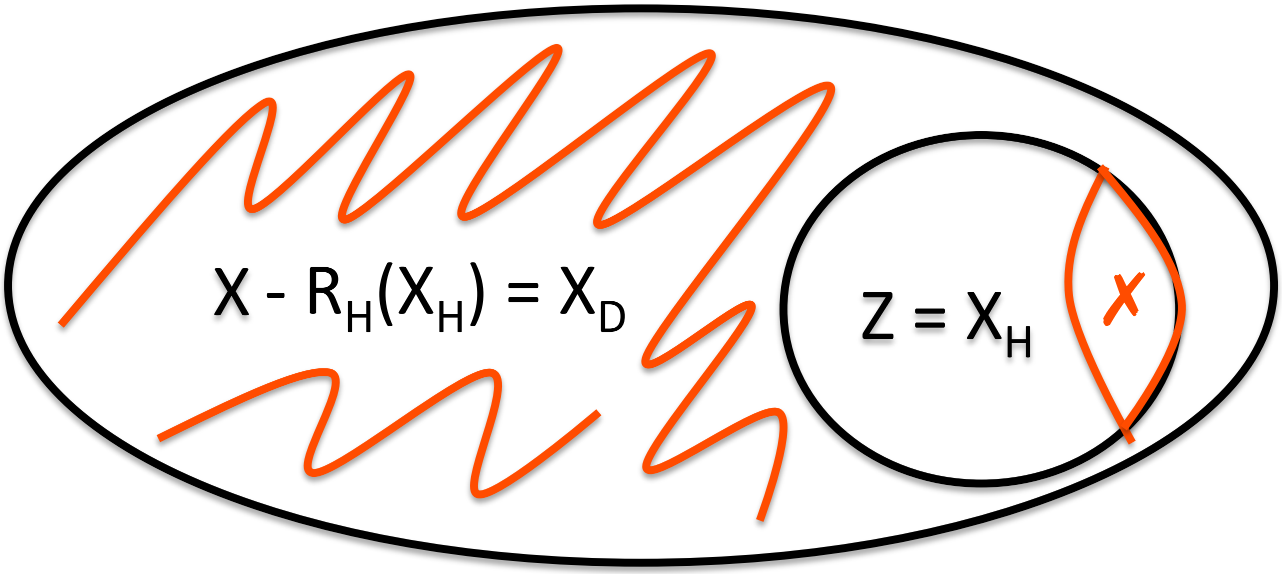
Set of contracts X' stable if

- feasible: each doctor appears at most once
- individually rational: $C_H(X') = C_D(X') = X'$
- no blocking coalitions: (h, X*) block X' if
  - $C_h(X') \neq X^*$
  - $C_h(X' \cup X^*) = X^*$
  - $C_D(X' \cup X^*)$ contains X*

Characterization of Stability

- X’ stable iff any alternative contract would be rejected by some doctor or hospital
- opportunity sets: currently considering Z
  - available to hospitals: $X - R_H(Z)$
  - available to doctors: $X - R_D(Z)$

Characterization of Stability



1. hospitals faced with options
2. hospitals reject some of their options
3. other options plus unproposed contracts are opportunities
4. when doctors face these opportunities, they must rule out unproposed contracts

Fixed Point Theorem

Theorem. If (X_D, X_H) is solution to

$$X_D = X - R_H(X_H)$$

$$X_H = X - R_D(X_D)$$

then the intersection of X_H and X_D is stable.

Conversely, if X' stable, there exist X_H and X_D whose intersection is X'.

Substitutability

Defn. Choice function $C_a(.)$ substitutable if for all contracts x, z and subsets X',

z not in $C_a(X' \cup \{z\})$ implies z not in $C_a(X' \cup \{x,z\})$

or, equivalently, $R_a(.)$ monotone.

Intuition: Receiving new offers makes agent weakly less-interested in old offers.

Substitutability

Example: contracts with wages: $X = D \times H \times W$

- demand theory substitutes: if at wage vector v_w, h chooses d, then h still chooses d at v'_w,
  - $v'_w(d) = v_w(d)$
  - $v'_w(d') \geq v_w(d')$ for all $d' \neq d$
- choice function substitutable iff satisfies demand theory substitutes

Tarski's Fixed Point Theorem

Defn. Given lattice (A, ≥), function f:A→A isotone if monotone on partial order:

$$a \geq b \text{ implies } f(a) \geq f(b)$$

Theorem. Any isotone function f on a complete lattice has a fixed point, and the set of fixed points form a complete lattice with respect to ≥.

Existence

- Define lattice $(X \times X, \geq)$ where

  $(X_D, X_H) \geq (Y_D, Y_H)$ iff $Y_D \subseteq X_D$ and $X_H \subseteq Y_H$
- Complete since $(X, \varnothing)$ and $(\varnothing, X)$ max/min elts
- Define $F(X_D, X_H) = (F_1(X_H), F_2(F_1(X_H)))$ where
  - $F_1(X') = X - R_H(X')$ and $F_2(X') = X - R_D(X')$
  - so $X' = X_H$ are contracts currently on the table
  - $F_1(X_H)$ everything hospitals don't reject
  - $F_2(F_1(X_H))$ what doctors pass back to hospitals

Existence

Claim. If prefs substitutable, then F isotone.

Prf. Follows from monotonicity of $R_h(.)$.

Interpretation. DA is repeated iterations of F,

- hospital-proposing, start at min elt ($X_D = \emptyset$, $X_H = X$)
- doctor-proposing, start at max elt ($X_D = X$, $X_H = \emptyset$)

Example: Hospital-Proposing

$P(d_1) = h_1, h_2$

$P(d_2) = h_1, h_2$

$P(h_1) = d_1, d_2$

$P(h_2) = \{d_1, d_2\}, d_1, d_2$

Initialize $X_D = \emptyset$, $X_H = X = \{(h_1, d_1), (h_1, d_2), (h_2, d_1), (h_2, d_2)\}$

- Hospitals reject: $R(X_H) = \{(h_1, d_2)\}$
- Hospitals offer: $X_D = \{(h_1, d_1), (h_2, d_1), (h_2, d_2)\}$
- Doctors reject: $R(X_D) = \{(h_2, d_1)\}$
- Doctors choose: $X_H = \{(h_1, d_1), (h_1, d_2), (h_2, d_2)\}$
- Hospitals reject: $R(X_H) = \{(h_1, d_2)\}$

Output $X_D \cap X_H = \{(h_1, d_1), (h_2, d_2)\}$.

Application: Unsplittable Flow

- jobs $J = \{j_1, \ldots, j_n\}$
  - job j has size s(j)
  - preference list P(j) over machines
- machines $M = \{m_1, \ldots, m_p\}$
  - machine m has capacity c(m)
  - preference list P(m) over individual jobs
  - preferences over sets responsive
- example: match groups of friends to sports teams without splitting up a group.

Application: Unsplittable Flow

- allocation μ assigns jobs to machines
  - strictly feasible: $\Sigma_{j \text{ in } \mu(m)} s(j) \leq c(m)$
  - stable: IR and no pair (j, m) s.t. $m >_j \mu(m)$ and for some j' in μ(m), $s(j) \leq c(m) - \Sigma_{k \neq j' \text{ in } \mu(m)} s(k)$, $j >_m j'$
- define $C_m(X)$ to be top contracts that "fit"
- then $C_m(X)$ substitutable, so stable allocations exist (must check defn's of stability match)

Application: Unsplittable Flow

- allocation μ assigns jobs to machines
  - weakly feasible: $\Sigma_{j \text{ in } \mu(m)} s(j) \leq c(m) + \max_j(s(j))$
  - stable: IR and no pair (j, m) s.t. $m >_j \mu(m)$ and for some j' in μ(m), $\Sigma_{k \neq j' \text{ in } \mu(m)} s(k) < c(m)$, $j >_m j'$
- now $C_m(X)$ is top contracts that just "overfit"
- by same argument, still substitutable
- stable allocations exist

Part 6: Large Market Results.

Entry-Level Labor Markets

Case Study:

National Residency Matching Program (NRMP): physicians look for residency programs at hospitals in the United States

A Brief History of NRMP

Case Study:

1950	1990	
decentralized, unraveling, inefficiencies	centralized clearinghouse, 95% voluntary participation	dropping participation sparks redesign to accommodate couples, system still in use

NRMP Theory and Practice

Theory:

Gale-Shapley stable marriage algorithm:
NRMP central clearinghouse algorithm
corresponds to deferred acceptance algorithm
(at first hospital-, and then student-proposing)

NRMP Redesign

What were the issues?

1. NRMP favored hospitals.
   Hospital-proposing deferred acceptance produces hospital-optimal matching.
2. NRMP was manipulable.
   Both students and hospitals have incentives to report false preferences.

Match Variations

Couples.
Married students have joint preferences over geographically close positions.

Reversion.
Hospital programs may wish to revert unfilled positions to other programs at same hospital.

Problems with Match Variations

What were the issues?

3. Algorithm choice affects unmatched agents.
   Not with no match variations (rural hospital theorem), but possible otherwise.
4. There may be no stable matching.
   Stable matching exists with no match variations, but may not otherwise.

Concerns with NRMP

1. NRMP favored hospitals.
2. NRMP was manipulable.
3. Algorithm choice affects unmatched agents.
4. There may be no stable matching.

… empirical study.

[Roth-Peranson '99]

Descriptive Statistics of NRMP

	1987	1993	1994	1995	1996
Applicants					
# with ROL	20071	20916	22353	22937	24749
# couples	694	854	892	998	1008
Programs					
# with ROL	3170	3622	3662	3745	3758
# positions	19973	22737	22801	22806	22578

Difference between DA Algorithms

	1987	1993	1994	1995	1996
# applicants affected	20	16	20	14	21
prefer hospital-proposing	8	0	9	0	9
prefer applicant-proposing	12	16	11	14	12
new matched	0	0	0	0	1
new unmatched	1	0	0	0	0

Difference between DA Algorithms

	1987	1993	1994	1995	1996
# programs affected	20	15	23	15	19
prefer hospital-proposing	12	15	11	14	9
prefer applicant-proposing	8	0	12	1	10
new matched	0	0	2	1	1
new unmatched	1	0	2	0	0

Bounding Potential Manipulations

Theorem. Equilibria produce stable matchings.

Corollary. An agent can manipulate only if he or she has more than one stable mate.

Bounding Potential Manipulations

Theorem. An agent's best stable mate is the one he or she receives when proposing (and worst when not proposing).

Corollary. An agent has more than one stable mate if and only if he or she receives different mates at the men-proposing and women-proposing algorithms.

Bounding Potential Manipulations

	1987	1993	1994	1995	1996
# applicants	20071	20916	22353	22937	24749
# applicants who could manipulate	20	16	20	14	21
# positions	19973	22737	22801	22806	22578
# programs	3170	3622	3662	3745	3758
# programs who could manipulate	20	15	23	15	19

Explanations

What limits number of stable mates?

1. Preferences are correlated.
   Applicants agree on prestigious hospitals; hospitals agree on promising applicants.
2. Preferences are short.
   Applicants typically list at most 15 hospitals.

A Probabilistic Model

Women: n hospital positions, preference is a uniform random permutation of all men

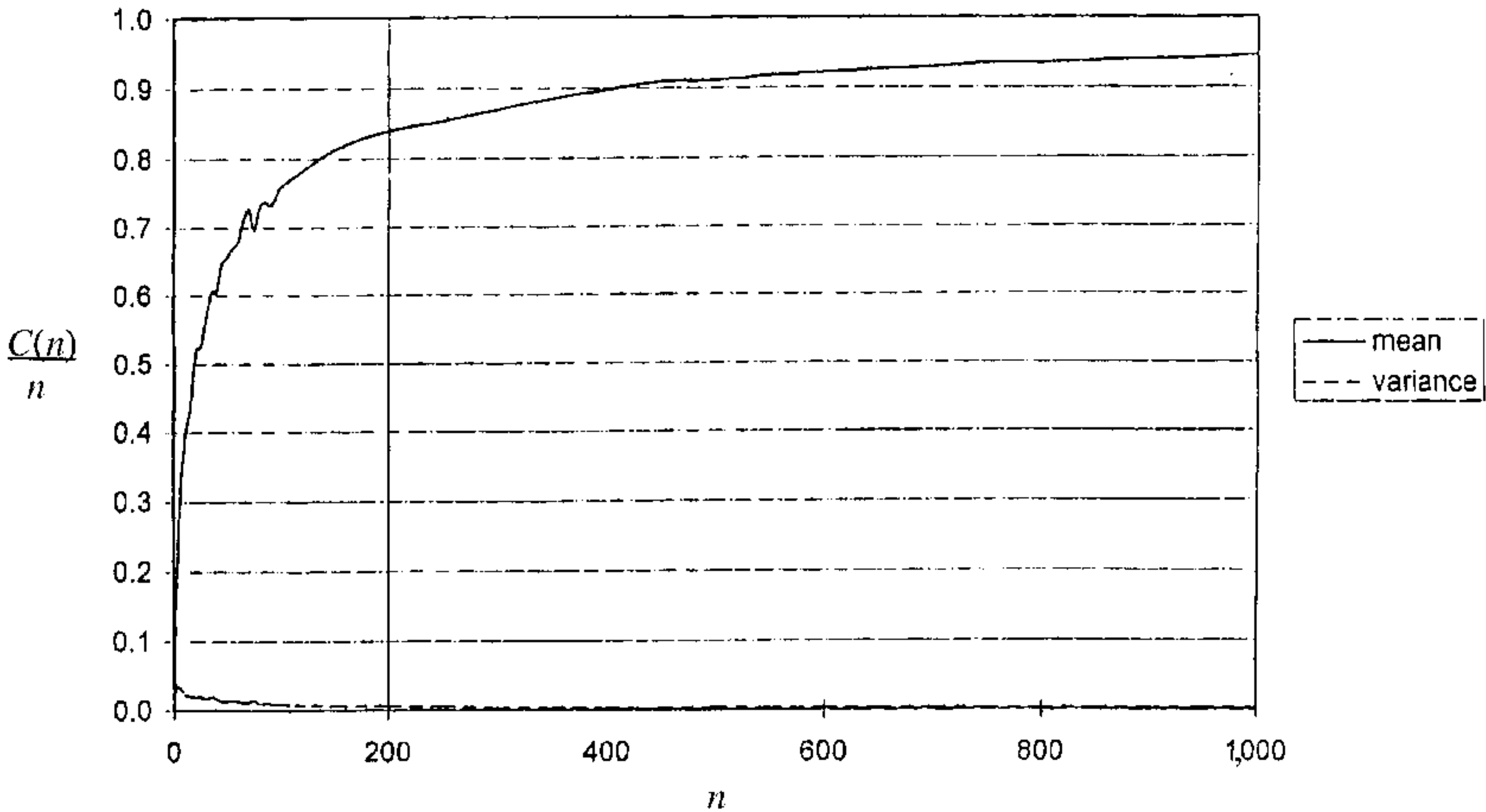
Men: n applicants, preference chosen uniformly at random from lists of at most k women

A Probabilistic Model

Conjecture [Roth-Peranson ‘99]. Holding k constant as n tends to infinity, the fraction of women with more than one stable mate tends to zero.

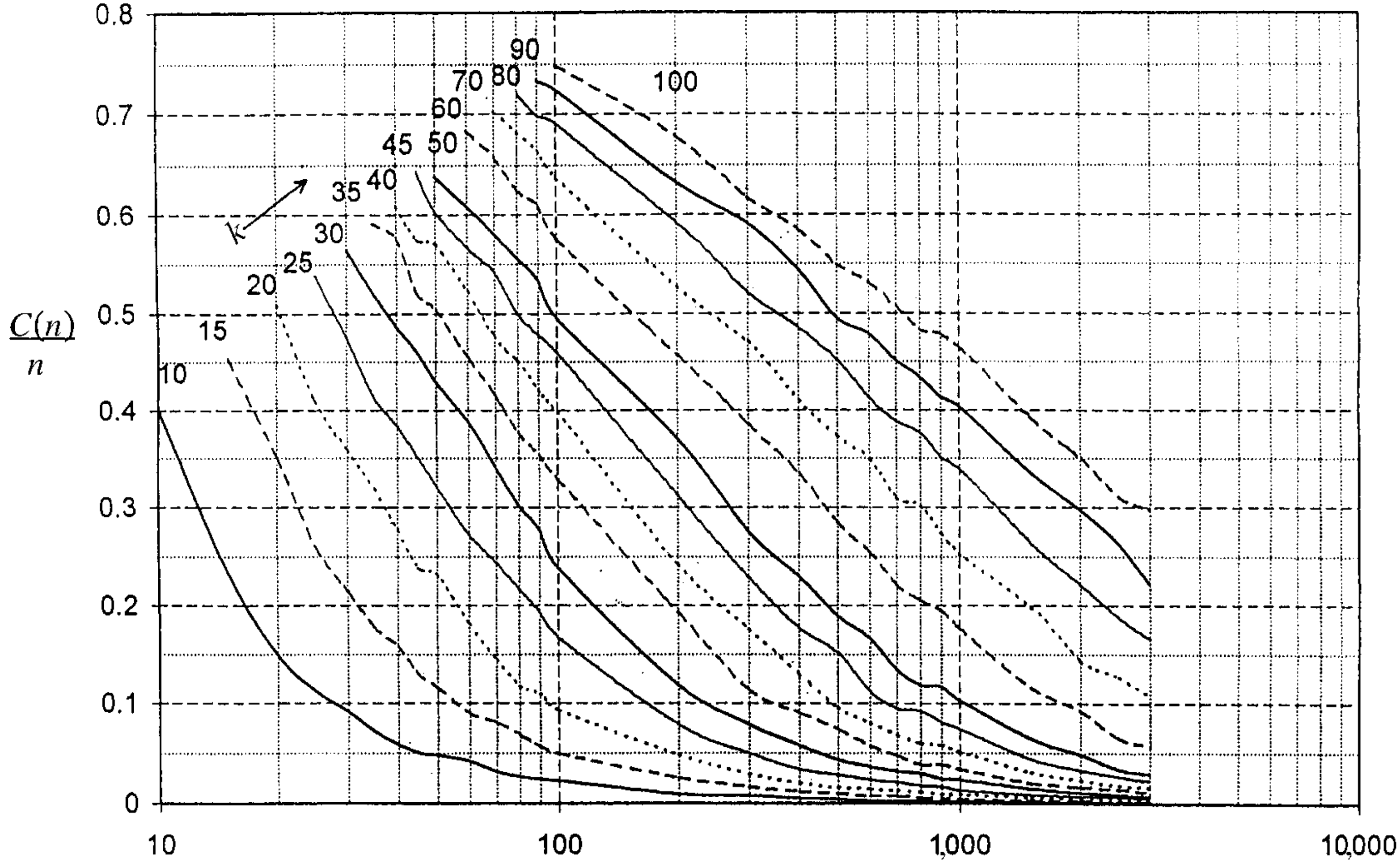
The potential to manipulate is vanishingly small.

Simulation



Fraction of agents with more than one stable mate when k = n.

Simulation



Fraction with more than one stable mate when k constant, n grows.

Theoretical Result

Theorem. Even allowing women *arbitrary* preferences, the fraction of agents with more than one stable mate tends to zero as n tends to infinity (holding k fixed).

[Immorlica-Mahdian ‘05, Kojima-Pathak ‘09]

Economic Implications

1. When others are truthful, almost surely an agent's best strategy is to tell the truth.
2. There is an equilibrium of women-proposing DA in which $(1-o(1)) \times n$ agents are truthful.
3. In settings of incomplete information, there is a $(1+o(1))$ approximate Bayes-Nash equilibrium in which all agents are truthful.