EECS 495: Combinatorial Optimization **Lecture 4**
Matching: Edmonds-Gallai Decomposition, Matching Polytope, TDI

Reading: Schrijver, Chapter 25

Recap

Theorem 0.1 *(Tutte-Berge Formula): For any graph G, $\nu(G) = min_{U \subseteq V}(|V| + |U| - o(G-U))/2$.*

Def: U is a Tutte-Berge witness if $\nu(G) = (|V| + |U| - o(G-U))/2$.

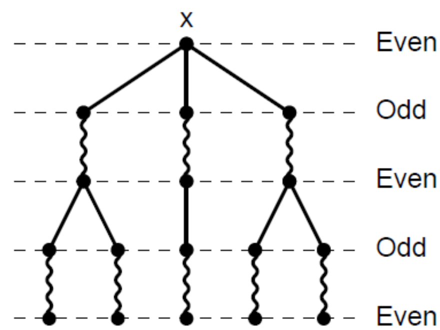
Def: The *Edmonds-Gallai decomposition* partitions the vertices V of a graph G into sets

- $D(G)$ – set of vertices v such that v is exposed by some maximum matching,
- $A(G)$ – set of neighbors of $D(G)$, and
- $C(G)$ – set of all remaining vertices.

Construction: vertices reachable by odd/even alternating paths from a vertex $v \in X$.

Let M be matching returned by Edmonds' Algorithm, X be exposed vertices.

- Even := $\{v : \exists$ even alternating path from X to $v\} = D(G)$, odd compoents in $G-U$ and factor critical
- Odd := $\{v : \exists$ odd alternating path from X to v and no even one$\} = A(G)$



- Free := $\{v : \nexists$ alternating path from X to $v\} = C(G)$, even components in $G-U$

Claim: There is no edge between Even and Free.

Claim: There is no edge within Even in G_0.

Claim: $C(G)$ is even components.

Proof: We proved no edge between Even and Free, so M matches vertices of $C(G)$ to vertices of $C(G)$ so $|M \cap E(C(G))| = |C(G)|/2$.

Claim: $D(G)$ is odd components, each of which is factor-critical.

Proof: For every connected component H of $(G-U) \cap D(G)$, we show:

1. Either $|X \cap H| = 1$ and $|M \cap \delta(H)| = 0$, or $|X \cap H| = 0$ and $|M \cap \delta(H)| = 1$ (where $\delta(H)$ is edges with exactly one endpoint in H).

2. H is factor-critical.

Tutte-Berge Witnesses

Theorem 0.2 *$U = A(G)$ is a Tutte-Berge witness.*

Proof: Want to show

$$|M| \geq \frac{1}{2}(|V| + |A(G)| - o(G \setminus A(G))$$

(other direction always holds). Note that

$$|M| \geq$$

$$|M \cap E(C(G))| + |M \cap E(D(G))| + |M \cap \delta(A(G))|$$

and

- we showed $|M \cap E(C(G))| = |C(G)|/2$
- previous proof, first subclaim, showed $|M \cap E(D(G))| = \frac{1}{2}(|D(G)| - o(G \setminus A(G)))$ (each component leaves one unmatched or matched to outside)
- $|M \cap \delta(A(G))| = |A(G)|$ since all $v \in A(G)$ matched to vertices of $D(G)$ (if not can grow matching)

so have

$$\frac{1}{2}\left(|C(G)| + |D(G)| + 2|A(G)| - o(G \setminus A(G))\right)$$

$$= \frac{1}{2}\left(|V| + |A(G)| - o(G \setminus A(G))\right)$$

as claimed.

Matching Polytope

Def: For a matching $M \subseteq E$, define its incidence vector $\chi(M) \in \Re^{|E|}$ to be $\chi(M)_e = 1$ if $e \in M$, 0 otherwise. The *matching polytope* $\mathcal{P}$ is the convex hull of incidence vectors of matchings.

Goal: Represent $\mathcal{P}$ by set of linear inequalities on variables $\{x_e\}$.

Question: Come up with some inequalities.

- $x_e \geq 0$
- $x(\delta(v)) = \sum_{e \in \delta(v)} x_e \leq 1$: each vertex has at most one adjacent edge

Call this polytope P_1.

Note: $\mathcal{P} \subseteq P_1$

Example: P_1 is not contained in $\mathcal{P}$: triangle

- $\mathcal{P} = conv\{(1,0,0), (0,1,0), (0,0,1), (0,0,0)\}$
- $(0.5, 0.5, 0.5) \in P_1$ but not in $\mathcal{P}$

Question: Additional constraint?

Def: The *blossom constraints* are

$$x(E(U)) = \sum_{e \in E(U)} x_e \leq \frac{|U| - 1}{2}, U \subseteq V, |U| \text{ odd.}$$

The polytop P_2 is P_1 together with the blossom constraints.

Theorem 0.3 *(Edmonds, 1965): P_2 equals the matching polytope $\mathcal{P}$.*

[[*Edmonds gave algorithmic proof, we use TDI.*]]

Total Dual Integrality

Recall primal/dual LPs:

Primal P:

$\max c^T x$ s.t. $Ax \leq b$

Dual D:

$\min b^T y$ s.t. $A^T y = c$ and $y \geq 0$

Def: A linear system $\{Ax \leq b\}$ is *totally dual integral* (TDI) if for any integral cost vector for the primal such that $\max c^T x, Ax \leq b$ is finite, there exists an integral optimal dual solution.

Theorem 0.4 *(Edmonds-Giles, 1979): If a system $\{Ax \leq b\}$ is TDI and b is integral, then $\{Ax \leq b\}$ is integral (i.e., the extreme points are integral).*

[[*We will prove this later.*]]

Note: We will show P_2 is TDI and hence is convex hull of all integral points contained in it, proving that $P_2 = \mathcal{P}$.

Polyhedral combinatorics:

- define $Ax \leq b$ and show integral with vertices corresponding to certain combinatorial objects.
- show system is TDI so dual has integral solution as well.
- find combinatorial interpretation for dual to get min-max theorem, or also helps design primal-dual algs by discretizing space.

[[*Rational polyhedra have TDI representations.*]]

Theorem 0.5 *(Giles-Pullyblank, 1979): For a rational polyhedron $\mathcal{P}$, there exist A and b with A integral such that $\mathcal{P} = \{x : Ax \leq b\}$ and the system is TDI.*

Note: b integral iff $\mathcal{P}$ integral

Example: $\mathcal{P} = conv\{(0,3),(2,2),(0,0),(3,0)\}$

Representation: $\{x, y : x \geq 0, y \geq 0, x + 2y \leq 6, 2x + y \leq 6\}$

Draw figure.

Suppose $c = (1,1)$. Primal opt is $(2,2)$ and tight constraints are $(1,2)$ and $(2,1)$.

[[*Tight constraints are of A, i.e., normals of facets at $(2,2)$.*]]

Thus for $A^T y = c$ to have integer solution, must be able to write c as *integer* combination of $(1,2)$ and $(2,1)$.

[[*Tight constraints in opt primal soln are non-zero variables in opt dual soln.*]]

Question: Make TDI with new representation?

Representation: add inequalities $x + y \leq 4, x, y, \leq 3$, becomes TDI.

Hilbert Basis

Question: When is a system TDI? Consider problem $\max\{cx : Ax \leq b\}$ with c integral and opt soln $\beta < \infty$.

- There's opt soln x^* in some face F defined by $\{Ax \leq b\}$ and $cx = \beta$.
- Suppose F is an extreme point, let $A'x \leq b'$ be inequalities tight at x^* (i.e., $A'x^* = b'$).
- Dual is $\min\{b^T y : A^T y = c, y \geq 0\}$ so opt dual corresponds to c being expressible as non-neg combination of row vectors, i.e., the *cone* of row vectors of A'.
- For y to be integral, must be able to ex-

press points in cone as integer combinations.

Def: A set of vectors $\{a_i : a_i \in \mathcal{Z}^n\}$ is a *Hilbert basis* if for any integral $c \in cone(a_i) = \{\sum_i \lambda_i a_i : \lambda_i \geq 0\}$, there exist non-negative integers μ_i such that $c = \sum_i \mu_i a_i$.

Example: For vertex $(3, 0)$ above, tight constraints $\{(1, 2), (-1, 0), (0, 1)\}$ form a Hilbert basis.

$\lambda_1 - \lambda_2 = c_1$ and $2\lambda_1 + \lambda_3 = c_2$ so for $\lambda_1 > 0$, $c_2/c_1 \geq 2$ and we can get all these. For $\lambda_1 = 0$, λ_2, λ_3 are non-neg integers if c integral, so we can get all these too.

Theorem 0.6 *The rational system $Ax \leq b$ is TDI iff for each face (actually sufficient to check for each extreme point), tight constraints form a Hilbert basis.*

$\left[\left[\right.\right.$*Follows by above observations, i.e., LP-duality.*$\left.\left.\right]\right]$

We can always add constraints to make it TDI:

Theorem 0.7 *Any rational polyhedral cone $C = \{\sum_i \lambda_i a_i : \lambda_i \geq 0, \lambda_i \in \mathcal{R}\}$ with $\{a_i\}$ integral has a finite integral Hilbert basis.*

Proof:

Let $Q = \{\sum_i \lambda_i a_i : 0 \leq \lambda_i \leq 1\}$ and note for any integral $c \in C$,

$$c = \sum_i \lambda_i a_i$$

$$= \sum_i (\lambda_i - \lfloor \lambda_i \rfloor) a_i + \sum_i \lfloor \lambda_i \rfloor a_i$$

Call this $z + w$. Note

- w integral since a_i and $\lfloor \lambda_i \rfloor$ are
- c integral by assumption hence z is too
- $z \in Q$
- $a_i \in Q$
- thus w integral combination of integral vectors in Q
- so $c = z + w$ is also integral combination of integral vectors in Q

and therefore $Q \cap \mathcal{Z}^n$ is a finite integral Hilbert basis for C.

Note: In fact don't need to assume $\{a_i\}$ integral, follows from rationality of cone.

[[*We are now ready to prove main theorem.*]]

Claim: (Edmonds-Giles, 1979): If a system $\{Ax \leq b\}$ is TDI and b is integral, then $\{Ax \leq b\}$ is integral.

Proof: By contradiction.

- Consider extreme point x^* of P s.t. $x_j^* \notin \mathcal{Z}$ for some j.
- Let c be integral vector s.t. x^* unique opt by picking rational vector in cone at x^* and scaling.
- Consider $\hat{c} = c + \frac{1}{q} e_j$ (inside cone for large enough q).
- Since $q\hat{c}^T x^* - qc^T x^* = x_j^* \notin \mathcal{Z}$, either $q\hat{c}^T x^*$ or $qc^T x^*$ not integral.
- By duality and fact that b is integral, one of corresponding dual soln $\hat{y}$ or y not integral.
- Contradicts TDI since both $q\hat{c}$ and qc integral.