EECS 495: Combinatorial Optimization **Lecture 2**
Matching: Edmonds-Gallai Decomposition, Edmonds' Algorithm

Reading: Schrijver, Chapter 24

Tutte-Berge Formula

For graph G and $U \subseteq V$,

- let $G - U$ be subgraph obtained by deleting vertices in U, and
- $o(G - U)$ be number of components of G that contain an *odd* number of vertices.

Theorem 0.1 *(Tutte-Berge Formula): For any graph G, $\nu(G) = min_{U \subseteq V}(|V| + |U| - o(G - U))/2$.*

Proof: Suppose G connected (formula's additive). Do induction on number of vertices.

Base case: one vertex, trivial.

Case 1: G contains vertex v covered by *all* maximum matchings (e.g., middle vertex in example).

- Then $\nu(G - \{v\}) = \nu(G) - 1$.
- By induction, Tutte-Berge Formula holds in $G - \{v\}$ for some set U'.
- Let $U = U' \cup \{v\}$. Then

$$\begin{aligned} \nu(G) &= \nu(G - v) + 1 \\ &= (|V - v| + |U - v| - o(G - v - (U - v)))/2 + 1 \\ &= (|V| - 1 + |U| - 1 - o(G - U))/2 + 1 \\ &= (|V| + |U| - o(G - U))/2. \end{aligned}$$

Case 2: for every vertex v there is a maximum matching M that does not cover v (e.g., 3-cycles).

Claim: Each maximum matching leaves exactly one vertex exposed.

Hence $\nu(G) = (|V| - 1)/2$ and Tutte-Berge Formula follows by choosing $U = \emptyset$.

Proof: (of claim): By contradiction: suppose each maximum matching leaves two vertices exposed.

Choose maximum matching M and two exposed vertices u and v such that distance $d(u, v) \geq 2$ is minimized over all choices of (M, u, v).

[[*Distance is at least 2 since if it's 1 we can add an edge contradicting maximality of M.*]]

Let t be intermediate vertex on shortest $u - v$ path and N a maximum matching that exposes it whose symmetric difference with M is minimal.

By minimality of (M, u, v), N must cover u and v, so there is some other vertex x that it does not cover which is covered by M.

Let y be vertex matched to x by M and note $y \neq t$ (otherwise could add to N).

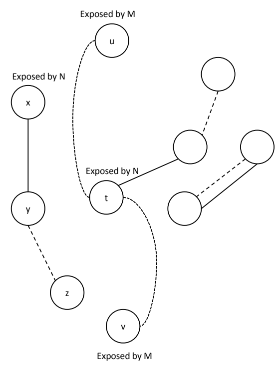
Let z be vertex matched to y by N and note $z \neq x$ (since x unmatched by N).

Then $N - yz + xy$ is a matching that exposes t and has smaller symmetric difference with

M contradicting choice of N.

□

Example: (for proof)



Edmonds-Gallai Decomposition

Def: A set U satisfying Tutte-Berge formula is called a *Tutte-Berge witness set.*

Note: Tutte-Berge witnesses give lots of information about matching. For example, vertices in a Tutte-Berge witness set are covered in every maximum matching.

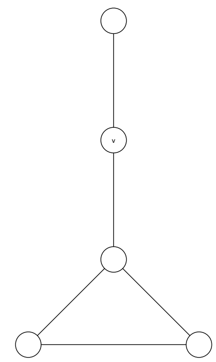
Question: Can we find a witness set U that satisfies formula?

Example: Tutte-Berge witness is not unique.

$U = \emptyset$ and $U = \{v\}$ are both witness sets.

[[*$U = \{v\}$ is more informative.*]]

Def: The *Edmonds-Gallai decomposition* partitions the vertices V of a graph G into sets



- $D(G)$ – set of vertices v such that v is exposed by some maximum matching,
- $A(G)$ – set of neighbors of $D(G)$, and
- $C(G)$ – set of all remaining vertices.

Theorem 0.2 *The set $U = A(G)$ is a Tutte-Berge witness. The set $D(G)$ contains all vertices in odd components in $G - U$ and $C(G)$ contains all vertices in even components in $G - U$.*

[[*U is an informative witness set in the sense that it describes all vertices that are in every maximum matching.*]]

Note: Can say more: each component of subgraph induced by $D(G)$ is *factor critical.*

Def: A graph H is *factor critical* if for any vertex v, $H - \{v\}$ contains a perfect matching.

[[*Hence in any odd component, we can choose which vertex gets left out.*]]

Example: Add an even cycle to figure, illustrate sets $D(G), A(G), C(G)$, show we can pick who to leave out from among odd components.

[[*We will get Edmonds-Gallai decomposition for free from Edmonds' algorithm for finding a maximum matching, which we describe next.*]]

Edmonds' Algorithm

Recall:

Def: An *alternating path* with respect to M is one that alternates edges.

Def: An *augmenting path* with respect to M is one that starts and ends at exposed vertices.

Claim: A matching M is maximum if and only if there are no augmenting paths with respect to M.

Algorithm: Finding a maximum matching:

- Start with empty matching.
- Repeatedly augment current matching along augmenting path if one exists.

[[*Same as for bipartite graphs but now harder to find augmenting paths.*]]

Flowers, stems, and blossoms

Given G construct directed graph $\hat{G}$ as follows:

- Vertices are V.
- There's an edge from u to w iff there's a v s.t. $(u, v), (v, w) \in E$, $(u, v) \notin M$, and $(v, w) \in M$.

Example: Two graphs, matching, corresponding directed graphs.

- line graph, length 3, middle edge in matching, and
- previous example, middle and bottom edge in matching, one extra exposed vertex at top connecting to middle vertex of line.

Note: Augmenting path in G corresponds to path in $\hat{G}$ that starts at an exposed vertex and ends at a neighbor of an exposed vertex. Converse not true due to odd cycles.

Idea: Eliminate these bad cycles by shrinking them.

Let X be set of exposed vertices.

Def: An *M-flower* is an M-alternating walk $v_0, \ldots, v_t$ (numbered so $(v_{2k-1}, v_{2k}) \in M$, $(v_{2k}, v_{2k+1}) \notin M$) such that:

- $v_0 \in X$,
- $v_0, \ldots, v_{t-1}$ distinct,
- $v_t = v_i$ for some even i,
- t is odd.

The portion from v_0 to v_i is the *stem*. The portion from v_i to v_t is the *blossom*.

[[*Below claim basically says flowers are our only problem.*]]

Claim: Let M be a matching in G and let $P = (v_0, \ldots, v_t)$ be a shortest alternating walk from X to X. Then either P is an M-augmenting path or $(v_0, \ldots, v_j)$ is a flower for some $j < t$.

Proof: If $v_0, \ldots, v_t$ all distinct, P is augmenting path. Else let j be smallest index such that $v_i = v_j$ for some $i < j$. Then this is a flower:

- $v_0 \in X$ by assumption,
- $v_0, \ldots, v_{j-1}$ distinct by choice of j,
- j is odd since otherwise $(v_{j-1}, v_j) \in M$ and then
  - if $i = 0$, $(v_{j-1}, v_j = v_i = v_0) \in M$ implies v_0 covered contradicting $v_0 \in X$, and

– if $0 < i < j-1$, then $(v_{j-1}, v_j = v_i) \in M$ contradicts that M is a matching since i is already matched to another vertex on alternating path.

- i is even since otherwise if i is odd (and j is too by previous item), then $(v_i, v_{i+1}), (v_j, v_{j+1}) \in M$ and so $v_{i+1} = v_{j+1}$ and so have cycle contradicting shortest alternating walk assumption (draw figure).

[[*So can use directing graph to find shortest alternating walk. If augmenting path, good, else if flower, shrink it as follows.*]]

Shrinking blossoms

Given flower $F = (v_0, \ldots, v_t)$ with blossom B, for any $v_j \in B$ can find matching M' s.t.

- every vertex of F is covered by M' except v_j
- M' agrees with M outside of F
- $|M'| = |M|$

by flipping edges of M along stem and taking matching in blossom that exposes v_j.

Def: (Shrinking a blossom) Given graph G and matching M with blossom B, the *shrunk graph* G/B with matching M/B is

- replace blossom with a single vertex b (think of this as stem base),
- eliminate self-loops/parallel edges,
- retain all non-deleted edges from M for M/B.

Example: Shrink blossom from initial figure with even cycle.

Theorem 0.3 *Let M be a matching of G and B be an M-blossom. Then M is maximum iff M/B is maximum in G/B.*

Proof: ($\rightarrow$): By contradiction.

- Suppose N/B matching in G/B larger than M/B.
- Define N in G by adding $\frac{1}{2}(|B|-1)$ edges of B to N/B.
- Then $|N| - |N/B| = \frac{1}{2}(|B|-1) = |M| - |M/B|$ (since B is M-blossom).
- Hence $|N| - |M| = |N/B| - |M/B|$ contradicting M maximum and M/B not.

($\leftarrow$): By direct proof.

- Suppose M is not maximum in G and let B be an M-blossom.
- Consider matching M' of equal cardinality which swaps edges of stem and note B is still an M'-blossom.
- Now M'/B leaves b exposed.
- Since M' not maximum, there's an augmenting path $v_0, \ldots, v_t$.
- Since B is M'-blossom, at most one endpoint of augmenting path is in B.
- Let $v_0 \notin B$ and v_i be first vertex of path in B ($v_i = v_t$ if path doesn't intersect B).
- Then $v_0, \ldots, v_{i-1}, b$ is augmenting path of M'/B in G/B, so M'/B not maximum.
- As M'/B same cardinality as M/B, we have M/B not maximum either.

□

Note: Lifting maximum matching N/B of G/B into matching N of G *does not* result in maximum matching.

Example: 3-cycle with a dangling edge off each vertex.

- Sub-opt matching M takes one edge of 3-cycle.
- Flower is 3-cycle (v_0 is exposed vertex, there's no stem).
- G/B is empty matching.
- Consider maximum matching N in G/B.
- Lifted to G, N takes one edge of 3-cycle to get size two, not maximum.

This is because N has no flowers.

Constructing maximum matchings

Algorithm: Initially M is maximal matching, X is exposed vertices of M, $\hat{G}$ is corresonding directed graph.

While $\hat{G}$ contains directed path from X to $N(X)$:

- Find such a path of minimum length.
- If corresponding path in G is augmenting, do augmentation.
- Else it's a flower so let B be blossom and G/B shrunk blossom.
  - Recursively find maximum matching N/B in G/B.
  - If $|N/B| = |M/B|$ then M maximum so return M.
  - Else unshrink N/B to get matching N with $|N| > |M|$ and repeat with $M := N$ and corresponding $X, \hat{G}$.

Analysis: Already proved correct. Running time:

- Compute $X, \hat{G}$: $O(m)$.
- Compute directed path in $\hat{G}$: $O(m)$ (BFS).
- Shrink blossom: $O(m)$.
- Depth of recursion: can shrink at most $O(n)$ times until terminating or augmenting matching.
- Can augment matching at most $O(n)$ times.

So alg is $O(mn^2)$.