Reading: Schrijver, Chapter 41

Matroid Intersection Algorithm

Claim: (Edmonds, 1970) For matroids M_1, M_2 on S,

$$\max_{J \in \mathcal{I}_1 \cap \mathcal{I}_2} \{|J|\} = \min_{A \subseteq S} \{r_1(A) + r_2(\overline{A})\}.$$

Strong exchange

Idea: Augmenting paths in graphs

Claim: Strong exchange property: For any B, B' bases, $\forall x \in B \setminus B', \exists y \in B' \setminus B$ such that $B - x + y$ and $B' - y + x$ are bases.

Def: *Span* of A is $\{e \in S | r(A + e) = r(A)\}$.

Claim:

1. If $A \subseteq B$, then $\text{span}(A) \subseteq \text{span}(B)$.
2. If $e \in \text{span}(A)$, then $\text{span}(A + e) = \text{span}(A)$.

Proof: By submodularity of r:

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$$

1. Suppose $e \in \text{span}(A)$ and let $X = A + e$, $Y = B$. Then,

   $$r(A + e) + r(B) \geq$$
   $$r((A + e) \cap B) + r(B + e) \geq$$
   $$r(A) + r(B + e)$$

   and $r(A) = r(A + e)$ so $r(B) \geq r(B + e)$. Equality by monotonicity, so $e \in \text{span}(B)$.

2. $\text{span}(A) \subseteq \text{span}(A + e)$ follows from 1. Other direction:
   - Suppose $e \in \text{span}(A)$ and $f \in \text{span}(A + e)$.
   - Let $x = A + e$ and $Y = A + f$.
   - Submodularity: $r(A + e) + r(A + f) \geq r(A + e + f) + r((A + e) \cap (A + f)) \geq r(A + e + f) + r(A)$.
   - $r(A) = r(A + e)$ and monotonicity imply $r(A + f) = r(A + e + f)$.
   - $f \in \text{span}(A + e)$ so $r(A + e + f) = r(A + e)$.
   - $e \in \text{span}(A)$ so $r(A + e) = r(A)$
   - thus $r(A + f) = r(A)$ so $f \in \text{span}(A)$.

Claim: (corollary): $\text{span}(\text{span}(A)) = \text{span}(A)$

Proof: (of strong basis exchange)

- suppose $x \in B \setminus B'$
- B' basis so $B' + x$ contains unique* circuit C
- C must contain x (subsets of B' indep) so $x \in \text{span}(C - x)$ so $x \in \text{span}((B \cup C) - x)$.

- by claim, $\text{span}((B \cup C) - x) = \text{span}(B \cup C) = S$ (since B basis)
- hence $B \cup C - x$ contains a basis B''
- $B - x$ and B'' both indep and $|B''| > |B - x|$ so $\exists y \in B'' \setminus (B - x)$ such that $(B - x) + y$ indep and a basis.
- and can pick y in $C - x$: $B'' \setminus (B - x) \subseteq ((B \cup C) - x) \setminus (B - x) \subseteq C - x$
- C unique circuit of $B' + x$ and $x, y \in C$ so $(B' - y) + x$ basis (contains no circuit since C unique)

*Why unique? Think graphic matroids...

Claim: If I indep and $I + x$ dependent, then $I + x$ contains a *unique* circuit.

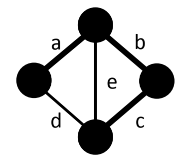
Proof: Suppose two circuits C_1 and C_2. Then $C_1 \cup C_2 - x$ contains a circuit:

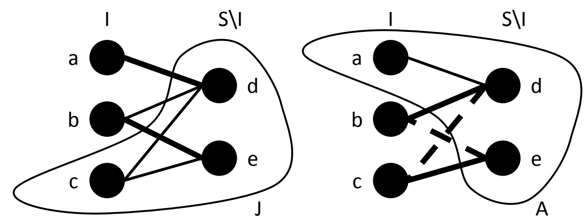
- Suppose $C_1 \cup C_2 - x$ indep
- Take $y \in C_1 \setminus C_2$
- Note $C_1 - y$ indep since C_1 circuit
- Extend to max indep set in $C_1 \cup C_2$, say Z
- Note Z doesn't contain C_2 (else C_2 indep) and doesn't contain y
- Hence $|Z| < |C_1 \cup C_2 - x|$ contradicting Z max indep set in $C_1 \cup C_2$.

Exchange graphs

Def: *Exchange graph* $\mathcal{D}(I)$ of M w.r.t. indep set I is bipartite graph with

- vertices $I, S \setminus I$





- edges (x, y) if $I - x + y$ indep

Claim: Let I and J be two indep sets with $|I| = |J|$. Then there's a perfect matching between $I \setminus J$ and $J \setminus I$ in $\mathcal{D}(I)$.

Proof:

- Consider *truncated* matroid $M' = (S, \{I' \in \mathcal{I} : |I'| \leq |I|\})$.
- Then I and J are bases in M'.
- Take $y \in J \setminus I$ and $x \in I \setminus J$ s.t. $I - x + y$ and $J - y + x$ bases in M' (x, y exist by strong basis exchange).
- Hence indep in M so (x, y) edge in $\mathcal{D}(I)$, add to matching
- replace I, J with $I, J - y + x$ and induct (as $I \setminus (J - y + x)$ has one less elt than $I \setminus J$

[[*Converse not true (why?), but...*]]

Claim: Let J be s.t. $|J| = |I|$ and $\mathcal{D}(I)$ has a *unique* perfect matching between $I \setminus J$ and $J \setminus I$. Then $J \in \mathcal{I}$.

Proof:

- Let N be unique matching
- Orient edges in N from $S \setminus I$ to I, rest from I to $S \setminus I$
- By uniqueness, no directed cycles
- Number vertices in topological sort of DAG $\mathcal{D}(I)$ s.t. $N = \{(y_1, x_1), \ldots, (y_t, x_t)\}$ and (x_i, y_j) never an edge for $i > j$
- Suppose J has a circuit C (for contradiction)
- Take largest i s.t. $y_i \in C$ (must exist one elt of C in $J \setminus I$ since $C \subseteq J$ and I indep)
- By choice of largest i, we have (x_i, y_j) not an arc for $y_j \in C - y_i$
- So $y_j \in \int\sqrt{\dashv}\backslash(I - x_i)$
- For all $z \in C - y_i$ either $z \in I \cap J$ or $= y_j \in J - I$ for some j, so $C - y_i \subseteq \text{span}(I - x_i)$
- But C a circuit, so $y_i \in \text{span}(C - y_i) \subseteq \text{span}(\text{span}(I - x_i)) = \text{span}(I - x_i)$ contradicting that $I - x_i + y_i \in \mathcal{I}$

Intersection exchange graph

[[*Overlay two copies of previous graph, direct M_1 edges left-to-right and M_2 edges right-to-left.*]]

Def: For $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, the *exchange graph* $\mathcal{D}_{M_1,M_2}(I) = (I, S \setminus I, E)$ with (y, x) an arc if $I - y + x \in \mathcal{I}_1$ and (x, y) an arc if $I - y + x \in \mathcal{I}_2$.

Let

- $X_1 = \{x \notin I | I + x \in \mathcal{I}_1\}$
- $X_2 = \{x \notin I | I + x \in \mathcal{I}_2\}$

Algorithm:

1. Find *augmenting path* P from X_1 to X_2 with no shortcuts. [[*If $X_1 \cap X_2 \neq \emptyset$, use singleton path.*]]
2. Replace I with $I \Delta P$
3. If no path ,set $U = \{z \in S | z \text{ can reach some vertex in } X_2\}$

Claim: Correctness:

1. When we stop, sets I and U give equality in min/max formula.
2. At each stage, $I \Delta P \in \mathcal{I}_1 \cap \mathcal{I}_2$.

Proof: (of 1): We show $r_1(U) = |I \cap U|$ (by similar proof $r_2(S \setminus U) = |I \setminus U|$).

- Note $X_2 \subseteq U$, $X_1 \cap U = \emptyset$, and U has no incoming arcs.
- If $r_1(U) \neq |I \cap U|$ then $r_1(U) > |I \cap U|$ (rank function monotone).
- Then $\exists x \in U \setminus (I \cap U)$ s.t. $(I \cap U) + x \in \mathcal{I}_1$.
- Note $I + x \notin \mathcal{I}_1$ since $X_1 \cap U = \emptyset$.
- Thus unique circuit in $I + x$ so $\exists y \in I \setminus U$ s.t. $I - y + x \in \mathcal{I}_1$ (killed only circuit in $I + x$).
- Then (y, x) arc in $\mathcal{D}_{M_1,M_2}(I)$ contradicting U has no incoming arcs.

Proof: (of 2): We show indep in M_1 (other similar).

- Let $P = x_0, y_1, x_1, \ldots, y_t, x_t$ be shortest path.
- Add new elt t to I indep of everything (in new matroids).

- Let $J = \{x_1, \ldots, x_t\} \cup (I \setminus \{y_1, \ldots, y_t\}$.
- Then $J \cup \{x_0\} \subseteq S \cup \{t\}$, $|J \cup \{x_0\}| = |I \cup \{t\}|$, and arcs from $\{y_1, \ldots, y_t\}$ to $\{x_1, \ldots, x_t\}$ plus arc from t to x_0 form unique perfect matching from $I \cup \{t\} \setminus J \cup \{x_0\}$ to $J \cup \{x_0\} \setminus I \cup \{t\}$ (since P has no shortcuts so y_1 has no other match, draw picture).
- Therefore $J \cup \{x_0\}$ indep in extended matroid so also in M_1.