Limitations of cross-monotonic cost-sharing schemes*

Nicole Immorlica† Mohammad Mahdian* Vahab S. Mirrokni‡

Abstract

A cost-sharing scheme is a set of rules defining how to share the cost of a service (often computed by solving a combinatorial optimization problem) amongst serviced customers. A cost-sharing scheme is cross-monotonic if it satisfies the property that everyone is better off when the set of people who receive the service expands. In this paper, we develop a novel technique for proving upper bounds on the budget-balance factor of cross-monotonic cost-sharing schemes. We apply this technique to games defined based on several combinatorial optimization problems including the problems of edge cover, vertex cover, set cover, and metric facility location, and in each case derive tight or nearly-tight bounds. In particular, we show that for the facility location game, there is no cross-monotonic cost-sharing scheme that recovers more than a third of the total cost. This result together with a recent 1/3-budget-balanced cross-monotonic cost-sharing scheme of Pál and Tardos closes the gap for the facility location game. For the vertex cover and set cover games, we show that no cross-monotonic cost-sharing scheme can recover more than a $O(n^{-1/3})$ and $O(\frac{1}{n})$ fraction of the total cost, respectively. Finally, we study the implications of our results on the existence of group-strategyproof mechanisms. We show that every group-strategyproof mechanism corresponds to a cost-sharing scheme that satisfies a condition weaker than cross-monotonicity. Using this, we prove that group-strategyproof mechanisms satisfying additional properties give rise to cross-monotonic cost-sharing schemes and therefore our upper bounds hold.

1 Introduction

Consider a situation where a group of customers (which we call *agents*) wish to buy a service such as connectivity to a network. The total cost of this service is a function of the group of customers that is serviced: a group of customers in distant towns might incur a larger cost than a group of customers in the same town. The service provider must develop a pricing policy, or *cost-sharing scheme*, that, given any group of customers, divides the cost of the service amongst them. For example, one plausible cost-sharing scheme divides the cost of the service evenly amongst the customers. However, in the case of network connectivity, this scheme seems to undercharge distant customers with high connection costs and overcharge other customers.

Developing a fair and economically viable cost-sharing scheme is a central problem in cooperative game theory (see, for example, [23] and [34]). The question of what constitutes an equitable cost-sharing is difficult to define and has been the subject of centuries of thought, dating from Aristotle's proclamation of "equal

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†Microsoft Research, Redmond, WA 98052, USA. Email: {nickle,mahdian}@microsoft.com. The first author was supported in part by an NSF fellowship. The second author was supported by a Microsoft fellowship.

‡Computer Science and Artificial Intelligence Laboratory, MIT, Cambridge, MA 02139, USA. Email: mirrokni@theory.csail.mit.edu. Research was supported in part by NSF contracts ITR-0121495 and CCR-0098018.

treatment of equals and unequal treatment of unequals in proportion to their inequality" in his book on Nicomachean Ethics [1] through modern times. One plausible notion of equity is that of *cross-monotonicity* or *population monotonicity* (see [32] for a survey). Intuitively, cross-monotonicity requires that the price charged to any individual in a group does not increase as the group expands. There is a large body of literature [5, 6, 12, 16, 22, 26, 29, 31] on cross-monotonic cost-sharing schemes for *submodular* cost functions[1], a subclass of cost functions of particular interest. Many mechanisms exist, prominent among them the Shapley value [29], which minimizes the worst-case efficiency loss, and the Dutta-Ray solution [6]. Both of these are budget-balanced and cross-monotonic for any submodular cost function.

There are many other interesting classes of cost functions that arise from (often NP-hard) optimization problems. For example, the cost of providing the service for a set S of agents could be expressed as the cost of building the cheapest Steiner tree that covers the elements of S, or the minimum cost of opening facilities and connecting each member of S to an open facility. These two games, and many others of practical import, are instances of covering problems. For such problems, it is usually impossible for a cross-monotonic cost-sharing scheme to be budget-balanced. Moreover, even if a budget-balanced cross-monotonic cost-sharing scheme exists, it might be hard to compute. Therefore, it is natural to consider cost sharing schemes that are *approximately budget balanced*, that is, they recover only a fraction of the cost of the service.[2] Approximately budget-balanced schemes have been proposed for minimum spanning tree [14, 17], Steiner tree [14], Steiner forest [18], facility location [25], and connected facility location [20].

We can derive simple bounds on the budget-balance factor of combinatorial optimization games using the integrality gaps of the "natural" LP-relaxations. The cross-monotonicity of a cost-sharing scheme implies that for every set of agents the cost shares form an allocation in the core of the game (see Section 2 for definitions). Therefore, the best budget-balance factor achievable by a cross-monotonic cost-sharing scheme cannot be better than that of a cost sharing in the core. A simple extension of the classic Bondareva-Shapley theorem [3, 28] implies that the best budget-balance factor for a cost sharing in the core of integer covering games is equal to the integrality gap of the "natural" LP-relaxation of the problem (this fact was observed by Jain and Vazirani [14]). This line of reasoning proves bounds on cross-monotonic cost-sharing schemes for many combinatorial optimization games. In particular, metric facility location, vertex cover, and set cover games cannot recover more than a $\frac{1}{1.463}$, $\frac{1}{2}$, and $\frac{1}{\ln n}$ fraction of the total cost, respectively. Prior to this work, this was the only method known for upper bounding the cross-monotonic cost-sharing schemes. In this paper, we show stronger upper bounds for several combinatorial optimization games using a novel technique based on the probabilistic method that will be explained in Section 3. In particular, we prove that the best budget-balance factor achievable for the facility location game is $\frac{1}{3}$, proving optimality of the scheme given by Pál and Tardos [25]. Also, for the vertex cover and set cover games, we show that no cross-monotonic cost sharing scheme can recover more than an $O(n^{-1/3})$ and $O(\frac{1}{n})$ fraction of the total cost, respectively. We also apply this technique to several other games including the maximum flow and the maximum matching games. In subsequent work, Könemann et al. [19] used our techniques to prove a tight bound of $\frac{1}{2}$ on the budget-balance factor of the Steiner tree game.

As observed by Moulin [22], cross-monotonic cost-sharing schemes can be used to construct *group strat-*

[1] Sometimes called *concave* games in the cooperative game theory literature.

[2] Alternatively, we can relax the definition of budget balance by allowing the scheme to recover at least the cost of the service and at most a small multiple of the cost of the service. This definition seems more reasonable, since a business usually needs to at least recover its costs. However, the two definitions are equivalent up to a constant multiple. To be consistent with other papers on this topic, we use the first definition in this paper.

egyproof mechanisms, or mechanisms which resist collusion among the agents. In fact, almost all known group-strategyproof mechanisms are constructed in this manner. However, as our results indicate, many classes of important cost functions fail to have budget-balanced cross-monotonic cost-sharing schemes. As we know that there are group-strategyproof mechanisms that do not correspond to any cross-monotonic cost-sharing scheme, our negative results for cross-monotonic schemes do not immediately imply negative results for group-strategyproof mechanisms. However, we give a partial characterization of group-strategyproof mechanisms in terms of cost-sharing schemes that satisfy a condition weaker than cross-monotonicity, and use this characterization to prove that group-strategyproof mechanisms that satisfy an additional condition called *upper continuity* give rise to cross-monotonic cost-sharing schemes, and therefore our negative results apply to such mechanisms.

The rest of this paper is organized as follows. In Section 2, we present the definitions of cross-monotonic cost-sharing schemes. Section 3 contains a description of our upper bound technique, highlighted by the example of the edge cover game (Section 3.1), and proof of bounds for the set cover game (Section 3.2), the vertex cover game (Section 3.3), the facility location game (Section 3.4) , and several combinatorial profit-sharing games (Section 3.5). In Section 4 we define group-strategyproof mechanisms and prove several results relating such mechanisms to cost-sharing schemes.

2 Definitions

Let $\mathscr{A}$ denote a set of n agents who are interested in a service. A *cost-sharing game* is defined by a function $C : 2^{\mathscr{A}} \mapsto \mathbb{R}^+ \cup \{0\}$ which for every set $S \subseteq \mathscr{A}$, gives the cost $C(S)$ of providing service to S.[3] A *cost allocation* for a set $S \subseteq \mathscr{A}$ is a function $\psi : S \mapsto \mathbb{R}^+ \cup \{0\}$, that for each agent $i \in S$, specifies the share $\psi(i)$ of i in the total cost of servicing S. A *cost-sharing scheme* is a collection of cost allocations for every $S \subseteq \mathscr{A}$.

Definition 2.1 *A* cost-sharing scheme *is a function* $\xi : \mathscr{A} \times 2^{\mathscr{A}} \mapsto \mathbb{R}^+ \cup \{0\}$ *such that, for every* $S \subset \mathscr{A}$ *and every* $i \notin S$, $\xi(i, S) = 0$*.*

Intuitively, we think of $\xi(i, S)$ as the share of i in the total cost if S is the set of agents receiving the service.

Ideally, we want cost-sharing schemes (and cost allocations) to be *budget-balanced*, that is, for every $S \subseteq \mathscr{A}$, $\sum_{i \in S} \xi(i, S) = C(S)$. Budget-balance is desirable as it guarantees economic viability of the auction. However, it is not always possible to achieve budget balance in combination with other properties, or even if it is possible, it might be computationally hard to compute the cost shares. Therefore, we relax this notion to the notion of α*-budget balance* (for some $\alpha \leq 1$).

Definition 2.2 *A cost-sharing scheme* ξ *is* α-budget-balanced *if, for every* $S \subseteq \mathscr{A}$, $\alpha C(S) \leq \sum_{i \in S} \xi(i, S) \leq C(S)$*.*

This definition guarantees that the mechanism does not over-charge agents, but it may under-charge them. Alternatively, one could define α-budget balance as $C(S) \leq \sum_{i \in S} \xi(i, S) \leq \frac{1}{\alpha} C(S)$ and equivalently relax

[3] This is similar to the notion of a *coalitional game with transferable payoff*, where the cost function is replaced by a function that gives the value, or the worth of each set. This notion was first defined by von Neumann and Morgenstern [33].

the notion of α-core (see Definition 2.3). All negative results hold without modification in this alternative framework as well; the positive results extend by multiplying each $\xi(i, S)$ by $\frac{1}{\alpha}$. To be consistent with other papers, we use the first definition in this paper.

In addition to budget balance, we usually require cost allocations and cost-sharing schemes to satisfy additional properties. One property that is extensively studied in the classic cooperative game theory literature [2, 3, 8, 27, 28, 30] is the property of being in the *core*, first suggested by Edgeworth [7] in 1881. This property intuitively says that no subset of agents should be overcharged for the service.

Definition 2.3 *A cost allocation ψ for a set $S \subseteq \mathscr{A}$ is in the α-*core *if and only if it is α-budget balanced and for every $T \subseteq S$, $\sum_{i \in T} \psi(i) \le C(T)$. A cost-sharing scheme ξ is in the α-core if and only if for every S, $\xi(\cdot, S)$ is in the α-core.*

Another property, which was studied by Moulin [22] and Moulin and Shenker [24] in order to design *group-strategyproof mechanisms* (see Section 4), and has recently received considerable attention in the computer science literature (see, for example, [14, 16, 17, 25]), is *cross-monotonicity* (or *population monotonicity*). This property captures the notion that agents should not be penalized as the serviced set grows. Namely,

Definition 2.4 *A cost-sharing scheme ξ is* cross-monotone *if for all $S, T \subseteq \mathscr{A}$ and $i \in S$, $\xi(i, S) \ge \xi(i, S \cup T)$.*

It is a simple exercise to show that every α-budget-balanced cross-monotonic cost-sharing scheme is in the α-core, but the converse need not hold. Therefore, cross-monotonicity is strictly stronger than the core condition. Using this fact and a simple extension of the classic Bondareva-Shapley theorem [3, 28] (see Jain and Vazirani [14]), one can derive upper bounds on the budget-balance factor of cross-monotonic cost-sharing schemes for covering games in terms of the integrality gap of their LP formulation. In the next section, we derive a technique based on the probabilistic method which yields stronger bounds.

3 Upper bounds for cross-monotonic cost-sharing schemes

In this section we present the main idea behind our upper bound technique and prove upper bounds for several games defined based on combinatorial optimization problems. We explain the technique in Section 3.1 with a simple example of the edge cover game and then extend it to the set cover game in Section 3.2. Sections 3.3, 3.4, and 3.5 contain the proofs of our bounds for the vertex cover, facility location, and several other games.

3.1 A simple example: the edge cover game

In this section, we explain our technique using the edge cover game as a guiding example. The edge cover game is defined as follows.

Definition 3.1 *Let $G = (V, E)$ be a graph with no isolated vertices. The set of agents in the edge cover game on G is the set of vertices of G. Given a subset S of vertices, the cost of S is the minimum size of a set $F \subseteq E$ of edges such that for every $v \in S$, at least one of the edges incident to v is in F. Such a set F is called an* edge cover *for S.*

It is easy to see that for every set S, one can obtain a minimum edge cover of S by taking a maximum matching on S and adding one edge for every vertex that is not covered by the maximum matching (see [4]). Using this fact, we can give a cost-sharing scheme that is in the $\frac{2}{3}$-core of the game: charge each vertex that is covered by the maximum matching $\frac{1}{3}$, and other vertices $\frac{2}{3}$. Since there is no edge between two vertices that are not covered by the maximum matching, this cost-sharing scheme satisfies the core property (but not cross-monotonicity). Furthermore, it is easy to see that the sum of the cost shares is always equal to $\frac{2}{3}$ times the edge cover for S. Therefore, there is a cost-sharing scheme satisfying the core property with a budget-balance factor of $\frac{2}{3}$. In fact, Goemans [9] showed that for every graph there is a cost-sharing scheme in the $\frac{3}{4}$-core. However, in the following, we show that no cross-monotonic cost-sharing scheme can achieve a budget-balance factor better than $\frac{1}{2}$.

Theorem 3.1 *For every $\epsilon > 0$, there is no $(\frac{1}{2} + \epsilon)$-budget balanced cross-monotonic cost-sharing scheme for the edge cover problem.*

Here is the high-level idea of the proof: We assume, for contradiction, that there is a cross-monotonic cost-sharing scheme that always recovers at least a $(\frac{1}{2} + \epsilon)$ fraction of the total cost. We explicitly construct a graph G (or in general the set of agents $\mathscr{A}$ and the structure based on which the cost function is defined), and look at the cost-sharing scheme on this graph. For edge cover, this graph is simply a complete bipartite graph $K_{n,n}$, with n large enough. Then, we need to argue that there is a set S of agents such that the total cost shares of the elements of S is less than $\frac{1}{2} + \epsilon$ times the size of the minimum edge-cover for S. This is done using the probabilistic method: we pick a subset S at random from a certain distribution and show that in expectation, the ratio of the recovered cost to the cost of S is low. Therefore, there is a manifestation of S for which this ratio is low. In the edge-cover example, we pick one vertex v of G uniformly at random and let S be the union of v and the set of vertices adjacent to v. We now need to bound the expected value of the sum of cost shares of the elements of S. We do this by using cross-monotonicity and bounding the cost share of each vertex $u \in S$ by the cost share of u in a substructure T_u of S. Bounding the expected cost share of u in T_u is done by showing that for every substructure T, every $u \in T$ has the same probability of occurring in a structure S in which $T_u = T$. This implies that the expected cost share of u in T_u (where the expectation is over the choice of S) is at most the cost of T_u divided by the number of agents in T_u. Summing up these values for all u gives us the desired contradiction.

Proof of Theorem 3.1. Assume that there is a $(\frac{1}{2} + \epsilon)$-budget-balanced cross-monotonic cost-sharing scheme ξ. Let G be the complete bipartite graph $K_{n,n}$, where n will be fixed later, and consider ξ on G. For every $v \in V(G)$, we let S_v be the union of v and the set of vertices adjacent to v (that is, all vertices of the other part). We pick a set S of agents by picking v uniformly at random from $V(G)$ and letting $S = S_v$. By the definition of the edge cover game,

$$C(S_v) = n \qquad \text{for every } v. \tag{1}$$

On the other hand,

$$\begin{aligned} \mathrm{E}_S\Big[\sum_{i \in S} \xi(i, S)\Big] &= \mathrm{E}_v\big[\xi(v, S_v)\big] + \mathrm{E}_v\Big[\sum_{u \in S_v \setminus \{v\}} \xi(u, S_v)\Big] \\ &\le 1 + \mathrm{E}_v\Big[\sum_{u \in S_v \setminus \{v\}} \xi(u, \{u, v\})\Big], \end{aligned} \tag{2}$$

where the last inequality follows from the facts that for every vertex u and every set S, $\xi(u,S) \le 1$, and that for every $v \in V(G)$ and $u \in S_v \setminus \{v\}$, $\xi(u,S_v) \le \xi(u,\{u,v\})$. Both of these facts are consequences of the cross-monotonicity of ξ. By the definition of expected values, we have

$$\mathrm{E}_v\Big[\sum_{u\in S_v\setminus\{v\}} \xi(u,\{u,v\})\Big] = n\,\mathrm{E}_{v,u}\big[\xi(u,\{u,v\})\big], \tag{3}$$

where the second expectation is over the choice of v from $V(G)$ and u in $S_v \setminus \{v\}$. However, choosing a vertex v and then a neighbor u of v at random is equivalent to choosing a random edge e in G at random, and letting u be a random endpoint of e and v be the other one. By the budget-balance condition, the sum of the cost shares of the endpoints of e is at most one. Therefore, for every e, if u is a random endpoint of e and v is the other endpoint, $\mathrm{E}[\xi(u,\{u,v\})] \le \frac{1}{2}$. Thus, the right-hand side of Equation 3 is at most $\frac{n}{2}$. Therefore, by Equations 1 and 2, we have

$$\mathrm{E}_S\left[\frac{\sum_{i\in S}\xi(i,S)}{C(S)}\right] \le \frac{1+\frac{n}{2}}{n} < \frac{1}{2}+\epsilon$$

for $n > 1/\epsilon$. Therefore, there is a set S satisfying $\frac{\sum_{i\in S}\xi(i,S)}{C(S)} < \frac{1}{2}+\epsilon$, which is a contradiction with the assumption that ξ is $(\frac{1}{2}+\epsilon)$-budget balanced. □

It is not difficult to see that the cost-sharing scheme ξ satisfying $\xi(i,S) = \frac{1}{2}$ for every $i \in S$ is cross-monotonic and $\frac{1}{2}$-budget balanced. Therefore, the bound given in the above theorem is tight.

3.2 The set cover game

The set cover game is defined as follows.

Definition 3.2 *Let $\mathscr{A}$ be a set of agents and $\mathscr{E}$ be a collection of subsets of $\mathscr{A}$ such that every element of $\mathscr{A}$ is contained in at least one set in $\mathscr{E}$. For every $S \subseteq \mathscr{A}$, the cost of S in the set cover game is the minimum size of a subcollection $\mathscr{F} \subseteq \mathscr{E}$ such that every $x \in S$ is contained in at least one set in $\mathscr{F}$. Such a collection $\mathscr{F}$ is called a* set cover *for S.*

One can think of the edge-cover problem as a special case of the set cover problem in which the size of each set is 2. It is not difficult to generalize Theorem 3.1 to the special case of set cover in which the size of each set is k, and prove that for k constant, no cross-monotonic cost-sharing scheme for this problem can recover more than a $\frac{1}{k}$ fraction of the cost. Using a similar argument, the next theorem shows that for the general case of the set cover game, no cross-monotonic cost-sharing scheme can recover more than a $O(\frac{1}{n})$ of the total cost.

Theorem 3.2 *There is no cross-monotonic cost-sharing scheme ξ for the set cover game such that for every set $S \subseteq \mathscr{A}$, ξ recovers more than a $O(\frac{1}{|S|})$ fraction of the cost of S.*

Proof. Assume that there is such a cross-monotonic cost-sharing scheme ξ. Consider the following set cover game. Let $\mathscr{A}$ be a set of n^2 agents that can be partitioned as $\mathscr{A} = A_1 \cup A_2 \cup \dots \cup A_n$, where A_i's are disjoint sets each of size n. Define $\mathscr{E}$ as the collection of all sets $S \subset \mathscr{A}$ such that $|S \cap A_i| = 1$ for every $i = 1, \ldots, n$. An alternative way to look at this is that $\mathscr{A}$ and $\mathscr{E}$ are sets of vertices and edges of an n-uniform n-partite complete hypergraph.

We pick a random set S of agents in the above game as follows: Pick a random i from $\{1, \ldots, n\}$, and for every $j \neq i$, pick an agent a_j uniformly at random from A_j. Let $T = \{a_j : j \neq i\}$ and $S = A_i \cup T$. The cost of the optimal set cover solution on S is always at least n, since no set in $\mathscr{E}$ contains two distinct elements of A_i, and therefore each element of A_i must be covered with a distinct set in $\mathscr{E}$.

We now bound the average recovered cost over the random choice of S.

$$\begin{aligned} \mathrm{E}_S\left[\sum_{x \in S} \xi(x, S)\right] &= \mathrm{E}\left[\sum_{x \in A_i} \xi(x, S)\right] + \mathrm{E}\left[\sum_{j \neq i} \xi(a_j, S)\right] \\ &\leq \mathrm{E}\left[\sum_{x \in A_i} \xi(x, \{x\} \cup T)\right] + \mathrm{E}\left[\sum_{j \neq i} \xi(a_j, T)\right] \end{aligned}$$

Since all elements of T can be covered by one set, the second term in the above expression is at most 1. We write the first term as $n\mathrm{E}_{S,x}\left[\xi(x, \{x\} \cup T)\right]$ where the expectation is over the random choice of S and the random choice of x from A_i. As in the proof of Theorem 3.5, the expected value of $\xi(x, \{x\} \cup T)$ in this experiment is equal to the expected value of $\frac{1}{n}\sum_{j=1}^{n} \xi(a_j, \{a_1, \ldots, a_n\})$ in an experiment that consists of choosing an agent a_j from each A_j uniformly at random. By the budget-balance property, we always have $\sum_{j=1}^{n} \xi(a_j, \{a_1, \ldots, a_n\}) \leq C(\{a_1, \ldots, a_n\}) = 1$. Therefore, the first term in the left-hand side of the inequality (4) is at most one. This means that the expected total cost share recovered from the set S is at most two. Therefore, the ratio of recovered cost to total cost of S is at most $2/n < 4/|S|$. □

It is worth noting that the above proof shows that even for the *fractional* set cover game, no cross-monotonic cost-sharing scheme can achieve a budget-balance factor better than $O(1/n)$.[4] This is particularly interesting for the following reason: It is easy to show that if there is an α-budget balanced cross-monotonic cost-sharing scheme for the fractional set cover, then for any special case of the set cover problem of integrality gap at most μ, there is an $\alpha\mu$-budget balanced cross-monotonic cost-sharing scheme. For example, if we could find a constant-factor for fractional set cover, we would automatically get a constant-factor for metric facility location, generalized Steiner tree, and many other network design games. Unfortunately, the above theorem shows this approach for designing cross-monotonic cost-sharing schemes fails to recover much of the cost.

3.3 The vertex cover game

The vertex cover game is defined on a graph $G = (V, E)$. The set of agents is the set of edges of G, and the cost of serving a set $S \subseteq E$ is equal to the minimum size of a set A of vertices such that for each $e \in S$, at least one of the endpoints of e is in A. Such a set is called a *vertex cover* for the set S. It is well-known that the integrality gap of the LP relaxation of vertex cover is 2, and therefore no allocation in core can recover more than half the cost of the solution in the worst case [3, 28]. We show in the following theorem that if we require the cost-sharing scheme to be cross-monotonic, then no constant-factor budget balanced scheme exists.

[4] Other bounds in the section also apply to the fractional variants of the corresponding games.

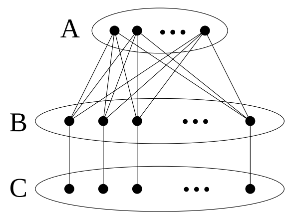

Figure 1: Vertex Cover Sample Distribution

Theorem 3.3 *For every $\epsilon > 0$, there is no cross-monotonic cost-sharing scheme for vertex cover that on every set S of n agents, recovers at least a $(2+\epsilon)n^{-1/3}$ fraction of the cost of S.*

Proof. Assume, for contradiction, that such a scheme ξ exists. We let G be a complete graph on $m + 2\ell$ vertices, where m and ℓ ($m < \ell$) are numbers that will be fixed later, and consider the cost-sharing scheme ξ on G. We show that there is some set S of edges of G for which ξ recovers at most a $|S|^{-1/3}$ fraction of the cost. We do this by picking S randomly from a distribution described below, and showing that the above statement holds in expectation, and therefore there should be a particular S satisfying the above statement.

Let π be a permutation of the $m + 2\ell$ vertices. Let A be the set of the first m vertices, B be the set of the next ℓ vertices, and C be the set of the remaining ℓ vertices. We denote the i'th vertices of B and C (based on the ordering given by π) by b_i and c_i. Let S_π denote the set of all $m\ell$ edges between A and B, together with the set of edges $b_i c_i$ for $i = 1, \ldots, \ell$. We pick S by picking the permutation π uniformly at random and letting $S = S_\pi$. See Figure 1 for an example.

If we denote the set of edges between A and B by T, we have

$$\mathrm{E}\Big[\sum_{e\in T}\xi(e,S)\Big] \le \mathrm{E}\Big[\sum_{e\in T}\xi(e,T)\Big] \le m, \tag{4}$$

where the first inequality follows from the cross-monotonicity of ξ and the second inequality is implied by the budget balance assumption and the fact that the cost of the minimum vertex cover in T is m. We also let T_i be the set of all $m + 1$ edges in S that have b_i as an endpoint (see Figure 1). Equation 4 and the cross-monotonicity of ξ imply the following.

$$\begin{aligned} \mathrm{E}_S\Big[\sum_{i\in S}\xi(i,S)\Big] &= \mathrm{E}\Big[\sum_{e\in T}\xi(e,S)\Big] + \sum_{i=1}^{\ell}\mathrm{E}\left[\xi(b_i c_i, S)\right] \\ &\le m + \sum_{i=1}^{\ell}\mathrm{E}\left[\xi(b_i c_i, T_i)\right], \end{aligned} \tag{5}$$

We now need to analyze the expectation of $\xi(b_i c_i, T_i)$ over the random choice of π. Notice that the only elements of π that are important in $\xi(b_i c_i, T_i)$ are the first m elements and the $m + i$'th and $m + \ell + i$'th elements (b_i and c_i). Therefore, the expectation of $\xi(b_i c_i, T_i)$ over the choice of π is equal to the expectation of $\xi(v_{m+2}v_{m+1}, \{v_1 v_{m+1}, v_2 v_{m+1}, \ldots, v_m v_{m+1}, v_{m+2} v_{m+1}\})$ over the random choice of an ordered list $v_1, v_2, \ldots, v_{m+2}$ of $m + 2$ different vertices of G. However, in this experiment it is clear by symmetry that the expected cost share of $v_i v_{m+1}$ is the same for $i = 1, \ldots, m, m + 2$, and therefore by the budget balance condition each of these expected cost shares is at most $\frac{1}{m+1}$. This, together with Equation 5 imply the following.

$$\mathrm{E}_S\left[\sum_{i\in S}\xi(i,S)\right]\le m+\frac{\ell}{m+1}. \tag{6}$$

On the other hand, the size of the minimum vertex cover in S is always ℓ. Therefore, the expected value of the ratio of $\sum_{i\in S}\xi(i,S)$ to $C(S)$ is at most $\frac{m}{\ell}+\frac{1}{m+1}$. Thus, there is a set S for which this ratio is at most $\frac{m}{\ell}+\frac{1}{m+1}$. Taking $m=\sqrt{\ell}$, we see that the allocation on S recovers at most a $\frac{2}{\sqrt{\ell}}<(2+\epsilon)|S|^{-1/3}$ fraction of the cost. □

We can show the following positive result for cross-monotonic cost sharing schemes for the vertex cover which, together with the Moulin mechanism [22] implies an approximately budget-balanced group-strategyproof mechanism for this problem (see Section 4). We do not know the right bound for the budget-balance factor of the vertex cover game.

Theorem 3.4 *For the vertex cover game, the cost-sharing scheme that charges the edge uv in the set S an amount equal to $\min(1/\deg_S(u),1/\deg_S(v))$ is cross-monotonic and $\frac{1}{2\sqrt{n}}$-budget balanced.*

Proof. It is clear that this scheme is cross-monotone. We only need to verify the budget-balance factor. Consider a set S of n agents (edges), and the graph $G[S]$ induced on this set of edges. We prove that the total cost share of the agents in S is at least $\frac{1}{2\sqrt{n}}$ times the cost of a vertex cover for S.

Divide the set of vertices into two subsets L and H, where L is the set vertices of degree less than $\sqrt{n}$ in $G[S]$ and H is the rest of vertices ($H=V(G)-L$). As a vertex cover solution, select H and both endpoints of all edges (u,v) such that $u,v\in L$. We show that the cost shares of the edges in S sum to at least a $\frac{1}{2\sqrt{n}}$ fraction of the cost of this solution. First consider any edge e between vertices in L. The cost share of e is at least $\frac{1}{\sqrt{n}}$, thus its cost share covers $\frac{1}{\sqrt{n}}$ of the cost of picking both its endpoints. Now consider the vertices in H. Since the degree of each vertex $v\in H$ is greater than or equal to $\sqrt{n}$, the sum of the cost shares of the edges adjacent to v is at least $\frac{1}{n}\sqrt{n}=\frac{1}{\sqrt{n}}$. Each edge is included in at most two such summations (namely, when both its endpoints are in H), and thus the sum of the cost shares of edges adjacent to vertices in H is at least a $\frac{1}{2\sqrt{n}}$ fraction of the cost of H. Therefore, the sum of the cost shares of the agents in S is at least $\frac{1}{2\sqrt{n}}$ times the cost of the optimal vertex cover for S. □

3.4 The metric facility location game

Given a set of cities, facilities with opening costs, and metric connection costs between cities and facilities, the facility location problem seeks to open a subset of facilities and connect each city to a facility in a manner that minimizes the total cost. In the facility location game, each city is an agent. The cost of a subset of agents is the cost of the minimum facility location solution for that subset; a cross-monotonic cost-sharing scheme tries to share this cost among the agents. In this section, we prove that any cross-monotonic cost-sharing scheme for facility location is at best $\frac{1}{3}$-budget-balanced. This matches the budget-balance factor of the scheme given by Pál and Tardos [25].

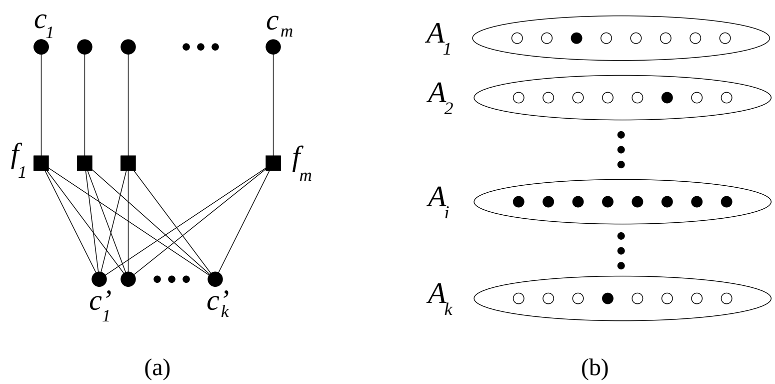

Figure 2: Facility Location Sample Distribution

We start by giving an example on which the scheme of Pál and Tardos [25] recovers only a third of the cost[5]. This example will be used as the randomly chosen structure in our proof.

Lemma 3.1 *Let $\mathcal{I}$ be an instance of the facility location problem consisting of $m + k$ cities $c_1, \ldots, c_m$, $c'_1, \ldots, c'_k$ and m facilities $f_1, \ldots, f_m$ each of opening cost 3. For every i and j, the connection costs between f_i and c_i and between f_i and c'_j are all 1, and other connection costs are obtained by the triangle inequality. See Figure 2(a). Then if $m = \omega(k)$ and k tends to infinity, the optimal solution for $\mathcal{I}$ has cost $3m + o(m)$.*

Proof. The solution which opens just one facility, say f_1, has cost $3m + k + 1 = 3m + o(m)$. We show that this solution is optimal. Consider any feasible solution which opens f facilities. The first opened facility can cover $k + 1$ clients with connection cost 1. Each additional facility can cover 1 additional client with connection cost 1. Thus, the number of clients with connection cost 1 is $k + f$. The remaining $m - f$ clients have connection cost 3. Therefore, the cost of the solution is $3f + k + f + 3(m - f) = 3m + k + f$. As $f \geq 1$, this shows that any feasible solution costs at least as much as the solution we constructed. $\square$

Theorem 3.5 *Any cross-monotonic cost-sharing scheme for the facility location game is at most $1/3$-budget balanced.*

Proof. Consider the following instance of the facility location problem. There are k sets $A_1, \ldots, A_k$ of m cities each, where $m = \omega(k)$ and $k = \omega(1)$. For every subset B of cities containing exactly one city from each A_i ($|B \cap A_i| = 1$ for all i), there is a facility f_B with connection cost 1 to each city in B. The remaining connection costs are defined by extending the metric, that is, the cost of connecting city i to facility f_B for $i \notin B$ is 3. The facility opening costs are all 3.

We pick a random set S of cities in the above instance as follows: Pick a random i from $\{1, \ldots, k\}$, and for every $j \neq i$, pick a city a_j uniformly at random from A_j. Let $T = \{a_j : j \neq i\}$ and $S = A_i \cup T$. See Figure 2(b) for an example. It is easy to see that the set S induces an instance of the facility location problem

[5]This example also shows that the dual computed by the Jain-Vazirani facility location algorithm [15] can be a factor 3 away from the optimal dual.

almost identical to the instance $\mathcal{I}$ in Lemma 3.1 (the only difference is that here we have more facilities, but it is easy to see that the only relevant facilities are the ones that are present in $\mathcal{I}$). Therefore, the cost of the optimal solution on S is $3m + o(m)$.

We show that for any cross-monotonic cost-sharing scheme ξ, the average recovered cost over the choice of S is at most $m+o(m)$ and thus conclude that there is some S whose recovered cost is at most $m+o(m)$. As in the previous proofs, we start bounding the expected total cost share by using the linearity of expectations and cross-monotonicity:

$$\begin{aligned}
\mathrm{E}_S\Big[\sum_{c\in S}\xi(c,S)\Big] &= \mathrm{E}\Big[\sum_{c\in A_i}\xi(c,S)\Big] + \mathrm{E}\Big[\sum_{j\neq i}\xi(a_j,S)\Big] \\
&\leq \mathrm{E}\Big[\sum_{c\in A_i}\xi(c,\{c\}\cup T)\Big] + \mathrm{E}\Big[\sum_{j\neq i}\xi(a_j,T)\Big]
\end{aligned}$$

Notice the set T has a facility location solution of cost $3+k-1$ and thus by the budget balance condition the second term in the above expression is at most $k+2$. The first term in the above expression can be written as $m\mathrm{E}_{S,c}\left[\xi(c,\{c\}\cup T)\right]$ where the expectation is over the random choice of S and the random choice of c from A_i. However, it can be seen easily that this is equivalent to the following random experiment: From each A_j, pick a city a_j uniformly at random. Then pick i from $\{1,\ldots,k\}$ uniformly at random and let $c = a_i$ and $T = \{a_j : j \neq i\}$. From this description it is clear that the expected value of $\xi(c,\{c\}\cup T)$ is equal to $\frac{1}{k}\sum_{j=1}^{k}\xi(a_j,\{a_1,\ldots,a_k\})$. This, by the budget balance property and the fact that $\{a_1,\ldots,a_k\}$ has a solution of cost $k+3$, cannot be more than $\frac{k+3}{k}$. Therefore,

$$\mathrm{E}_S\Big[\sum_{c\in S}\xi(c,S)\Big] \leq m(\frac{k+3}{k}) + (k+2) = m + o(m), \tag{7}$$

when $m = \omega(k)$ and $k = \omega(1)$. Therefore, the expected value of the ratio of recovered cost to total cost tends to $1/3$. □

3.5 Other combinatorial optimization games

In this section we prove bounds for three other combinatorial optimization games (in particular, the ones considered by Deng, Ibaraki, and Nagamochi [4]). These problems are maximization problems; therefore instead of cost-sharing schemes, we consider *profit-sharing* schemes, as defined below.

Definition 3.3 *A* profit-sharing game *(or a* coalitional game with transferable utilities*) is defined by a set $\mathscr{A}$ of agents, and a function $v : 2^{\mathscr{A}} \mapsto \mathbb{R}^+ \cup \{0\}$ that for every set S, gives the value $v(S)$ of S (or the profit earned if agents in S collaborate). A profit-sharing scheme is a function $\xi : \mathscr{A} \times 2^{\mathscr{A}} \mapsto \mathbb{R}^+ \cup \{0\}$, such that for every $S \subseteq \mathscr{A}$ and every $i \notin S$, $\xi(i,S) = 0$. Such a scheme is called α-*budget-balanced *(for some $\alpha \geq 1$) if for every $S \subseteq \mathscr{A}$, $v(S) \leq \sum_{i\in S}\xi(i,S) \leq \alpha v(S)$. A profit-sharing scheme ξ is in the α-*core* if it is α-budget-balanced and for every S and $T \subseteq S$, $\sum_{i\in T}\xi(i,S) \geq v(T)$. A profit-sharing scheme ξ is* cross-monotone *if for all $S, T \subseteq \mathscr{A}$ and $i \in S$, $\xi(i,S) \leq \xi(i,S\cup T)$.*

In this section, we consider profit-sharing schemes for the games of maximum flow, maximum arborescence packing, and maximum matching, and derive lower bounds on the budget-balance factor of cross-monotonic profit-sharing schemes for these games.

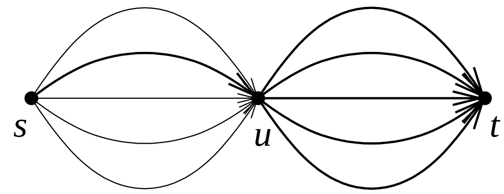

Figure 3: The graph G for the maximum flow game

The maximum flow game In the maximum flow game, we are given a directed graph $G = (V, E)$ with a source s and a sink t. Agents are directed edges of G. Given a subset of edges, S, the value of S is the value of the maximum flow from s to t on the subgraph of G induced by the edges of S. It is known that the core of the maximum flow game is nonempty [4]. The situation is different for cross-monotonic profit-sharing schemes.

Theorem 3.6 *There is no $o(n)$-budget-balanced profit-sharing scheme for the maximum flow game where n is the number of agents in the set that receives the service.*

Proof. Let G be a graph consisting of three nodes named s, u, and t; $n-1$ edges from s to u; and $n-1$ edges from u to t. Let E_{su} and E_{ut} denote the set of edges from s to u and from u to t, respectively. See Figure 3. We pick a random set S of n agents as follows: With probability $1/2$, pick a random edge e from s to u, and let $S = \{e\} \cup E_{ut}$. With probability $1/2$, pick a random edge e from u to t, and let $S = \{e\} \cup E_{su}$. For example the set S could contain the thick edges in Figure 3.

Assume ξ is an $o(n)$-budget-balanced cross-monotonic profit-sharing scheme for G. We have

$$\begin{aligned}
\mathrm{E}_S\left[\sum_{a\in S}\xi(a,S)\right] &\geq \frac{1}{2}\mathrm{E}_{e\overset{R}{\leftarrow}E_{su}}\left[\sum_{a\in E_{ut}}\xi(a,\{e\}\cup E_{ut})\right] + \frac{1}{2}\mathrm{E}_{e\overset{R}{\leftarrow}E_{ut}}\left[\sum_{a\in E_{su}}\xi(a,\{e\}\cup E_{su})\right] \\
&\geq \frac{1}{2}\mathrm{E}_{e\overset{R}{\leftarrow}E_{su}}\left[\sum_{a\in E_{ut}}\xi(a,\{a,e\})\right] + \frac{1}{2}\mathrm{E}_{e\overset{R}{\leftarrow}E_{ut}}\left[\sum_{a\in E_{su}}\xi(a,\{a,e\})\right] \\
&= (n-1)\mathrm{E}_{a\overset{R}{\leftarrow}E_{su},b\overset{R}{\leftarrow}E_{ut}}\left[\frac{1}{2}\xi(a,\{a,b\}) + \frac{1}{2}\xi(b,\{a,b\})\right] \\
&\geq \frac{n-1}{2}.
\end{aligned}$$

On the other hand, the value of every set S picked using the above procedure is one. Therefore, the expected ratio of the sum of profit shares to the value of S is at least $(n-1)/2$. $\square$

Remark 3.1 *It is easy to see that the above proof also works for the problems of packing the maximum number of arborescences in a digraph, and gives the same lower bound. An r-arborescence is a spanning tree rooted at r in which all edges are directed away from r. The maximum r-arborescence game is defined on a digraph $G = (V, E)$ with a root r where each edge is an agent. The value of a set S is the maximum* *number of edge-disjoint r-arborescences on the subgraph induced by S. One can think of the value of S as the maximum bandwidth for broadcasting messages from r to all vertices of the graph. It is known that the core of this game is nonempty [4].*

The maximum matching game As a last example, we consider the maximum matching game, in which the agents are vertices of a graph G, and the value of a subset of vertices S is the size of the maximum matching in the subgraph of G induced by S (denoted $G[S]$). One can show that there is a 2-budget-balanced profit-allocation in the core of this game.

Theorem 3.7 *There is no $o(n)$-budget-balanced profit-sharing scheme for the maximum matching game, where n is the set of agents that receive the service.*

Proof. We use the same construction that was used in the proof of Theorem 3.1. Let G be a complete bipartite graph with $n-1$ vertices in each part (here we use $n-1$ instead of n so that the size of S becomes n), and pick S by picking a random vertex in G and all vertices in the other part. Using an argument essentially the same as the one in the proof of Theorem 3.1, the expected sum of profit shares of the elements of S is at least $(n-1)/2$. On the other hand, the value of S is always one. Thus, there is an S on which the ratio between the total profit share and the value of S is at least $(n-1)/2$. $\square$

4 Group-strategyproof mechanisms

One of the important applications of cross-monotonic cost-sharing schemes is in the construction of group-strategyproof cost-sharing mechanisms [22, 24]. In this section, we explore the connection between cross-monotonic cost-sharing schemes and group-strategyproof cost-sharing mechanisms, and implications of the upper bounds of the previous section on such mechanisms. In Section 4.1 we define the setting and present some preliminaries. In Section 4.2 we discuss an issue in the definition of group-strategyproof mechanisms, and note that in order to exclude a trivial mechanism, we need to use a stronger version of one of the axioms. In Section 4.3 we give a partial characterization of group-strategyproof mechanisms in terms of cost-sharing schemes satisfying a property weaker than cross-monotonicity. We then use this characterization to prove that group-strategyproof mechanisms that satisfy additional properties give rise to cross-monotonic cost-sharing schemes.

4.1 Preliminaries

Let $\mathscr{A}$ be a set of n agents interested in receiving a service. Each agent i has a value $u_i \in \mathbb{R}$ for receiving the service, that is, she is willing to pay at most u_i to get the service. We further assume that the utility of agent i is given by $u_i q_i - x_i$, where q_i is an indicator variable which indicates whether she has received the service or not, and x_i is the amount she has to pay. A *cost-sharing mechanism* is an algorithm that elicits a bid $b_i \in \mathbb{R}$ from each agent, and based on these bids, decides which agents should receive the service and how much each of them has to pay. More formally, a cost-sharing mechanism is a function that associates to each vector $\mathbf{b}$ of bids a set $Q(\mathbf{b}) \subseteq \mathscr{A}$ of agents to be serviced, and a vector $x(\mathbf{b}) \in \mathbb{R}^n$ of payments.

When there is no ambiguity, we write Q and x instead of $Q(\mathbf{b})$ and $x(\mathbf{b})$, respectively. We assume that a mechanism satisfies the following conditions:[6]

- *No Positive Transfer (NPT)*: The payments are non-negative (that is, $x_i \geq 0$ for all i).
- *Voluntary Participation (VP)*: An agent who does not receive the service is not charged (that is, $x_i = 0$ for $i \notin Q$), and an agent who receives the service is not charged more than his bid (that is, $x_i \leq b_i$ for $i \in Q$)
- *Consumer Sovereignty (CS)*: For each agent i, there is some bid b_i^* such that if i bids b_i^*, she will get the service, no matter what others bid.

Furthermore, we would like the mechanisms to be approximately budget balanced. Mimicking the definition for cost-sharing schemes, we call a mechanism α-budget balanced if the total amount the mechanism charges the agents is between $\alpha C(Q)$ and $C(Q)$ (that is, $\alpha C(Q) \leq \sum_{i \in Q} x_i \leq C(Q)$).

We look for mechanisms, called *group strategyproof mechanisms*, which satisfy the following property in addition to NPT, VP, and CS. Let $S \subseteq \mathscr{A}$ be a coalition of agents, and u, u' be two vectors of bids satisfying $u_i = u'_i$ for every $i \notin S$ (we think of u as the value of agents, and u' as a vector of strategically chosen bids). Let (Q, x) and (Q', x') denote the outputs of the mechanism when the bids are u and u', respectively. A mechanism is *group strategyproof* if for every coalition S of agents, if the inequality $u_i q'_i - x'_i \geq u_i q_i - x_i$ holds for every $i \in S$, then it holds with equality for every $i \in S$. In other words, there should not be any coalition S and vector u' of bids such that if members of S announce u' instead of u (their value) as their bids, then every member of the coalition S is at least as happy as in the truthful scenario, and at least one person is happier.[7]

Given a cross-monotonic cost-sharing scheme ξ, Moulin [22] defined a cost-sharing mechanism $\mathcal{M}_\xi$ as follows.

Mechanism $\mathcal{M}_\xi$:

Initialize $S \leftarrow \mathscr{A}$.
Repeat
    Let $S \leftarrow \{i \in S : b_i \geq \xi(i, S)\}$.
Until for all $i \in S$, $b_i \geq \xi(i, S)$.
Return $Q = S$ and $x_i = \xi(i, S)$ for all i.

Notice that the mechanism $\mathcal{M}_\xi$ always services the maximal subset of agents whose bids are all at least as large as their cost shares in that set.[8] Moulin [22] proved the following result.

Theorem A (Moulin [22]) *If ξ is a cross-monotonic cost-sharing scheme, then $\mathcal{M}_\xi$ is group-strategyproof.*

[6]For a discussion about these properties see Moulin [22] and Moulin and Shenker [24].

[7]Notice that we do not allow members of the coalition to sacrifice their own utility to benefit the group's total utility, that is we disallow side-payments. Side-payments require a transfer of money between agents which might be restricted in some settings either due to legal concerns or issues of trust, and so we do not consider side-payments here. For a discussion of collusion with side-payments, see Goldberg and Hartline [11].

[8]Note that there is a unique maximal set as if two sets are feasible then, by cross-monotonicity, their union is as well.

4.2 A discussion about the definition

In the definition of group-strategyproof mechanisms in the paper by Moulin and Shenker [24] (which is the basis for the definition of this concept in most computer science papers), it is not required that an agent can bid in a way that guarantees her not to receive the service. In particular, it is assumed that the bids are non-negative, and an agent who bids zero can still be serviced, if her payment is also zero [24, page 517]. As we see in the following example, according to this definition, for every cost function there is a trivial budget-balanced group-strategyproof mechanism.

Example 4.1 *Arbitrarily order the agents from 1 to n. Then, find the first agent i in this order whose bid is at least $C(\{i, \ldots, n\})$. The set that will receive the service is $Q = \{i, \ldots, n\}$, and the total cost of servicing this set is paid by the agent i. Other agents pay nothing.*

Proposition 4.1 *Assuming non-negative bids, the mechanism in Example 4.1 is budget-balanced and group-strategyproof.*

Proof. It is not hard to see that this mechanism is budget-balanced and satisfies NPT, VP, and CS. To show that it is group-strategyproof, let i be the first agent to receive service when agents bid truthfully (or $n + 1$ if no agent receives service) and j be the first agent to receive service when a coalition deviates. If $j < i$, it must be that j is part of the coalition and raised his bid to a number greater than or equal to $C(\{j, \ldots, n\})$, but this decreases his utility. If $j = i$, then the outcome is identical to the truthful scenario and so no utility changes. If $j > i$, then the utility of any agent $k < j$ is now zero and so did not increase. The utility of any agent $k > j$ did not change as his allocation and payment remained the same. Finally, as the payment of j is at least his payment in the truthful scenario, the utility of agent j can not increase either. Thus the coalition can not be successful. $\square$

Although it satisfies all of the axioms, this mechanism is unsatisfactory, since in practice a coalition can convince a member that has zero utility for receiving the service simply not to bid, thus reducing the cost to others. Furthermore, this mechanism fails to satisfy the axioms in the original paper of Moulin [22], where a stronger version of CS is assumed that guarantees that each agent can bid in a way that she does not receive the service, no matter how others bid.

In order to exclude mechanisms like the one in Example 4.1, we only consider mechanisms that satisfy the stronger definition of CS by Moulin [22]. To this end, we allow the utilities and bids to be *negative*. NPT and VP guarantee that any agent with negative bid will not receive the service. An alternative approach (adapted by Moulin [22]) is to assume that utilities, bids, and payments are all positive.[9] In many combinatorial games, the cost function is not strictly increasing and therefore it is reasonable to allow cost shares to be zero. Thus, we use negative bids to indicate that an agent does not want to receive the service. However, it is easy to see that all our results hold in the setting considered by Moulin [22].

4.3 A partial characterization of group-strategyproof mechanisms

In Section 3, we proved that for certain games every cross-monotonic cost-sharing scheme is poorly budget balanced. A natural question to ask is whether all group-strategyproof mechanisms for these games are so

[9] This is equivalent to a property called *no free riders*, or *no free lunch*, which was used in an earlier version of this paper [13].

poorly budget balanced. Towards this aim, one might hope to show a converse to Theorem A, namely that every group-strategyproof mechanism corresponds to a cross-monotonic cost-sharing scheme. Unfortunately, this statement is not necessarily true (See, for example, Appendix A, or the incremental cost-sharing method for supermodular cost functions in the paper by Moulin [22]). In this section, we prove that for any group-strategyproof mechanism, we can construct a cost-sharing scheme that satisfies a weaker condition than cross-monotonicity. Then, we use this characterization to show that group-strategyproof mechanisms that satisfy certain additional properties correspond to cross-monotonic cost-sharing schemes.

We start by defining a property weaker than cross-monotonicity for cost-sharing schemes. Recall that a cost-sharing scheme is cross-monotonic, if the removal of each agent from the service set does not increase the cost to any other agent.

Definition 4.1 *Let $\xi : \mathscr{A} \times 2^{\mathscr{A}} \mapsto \mathbb{R}^+ \cup \{0\}$ be a cost-sharing scheme, $S \subseteq \mathscr{A}$, and $i \in S$. We say i is a* positive *element of S if for every $j \in S \setminus \{i\}$, $\xi(j, S \setminus \{i\}) \geq \xi(j, S)$ and for at least one such j a strict inequality holds; i is a* negative *element of S if for every $j \in S \setminus \{i\}$, $\xi(j, S \setminus \{i\}) \leq \xi(j, S)$ and for at least one such j a strict inequality holds. If for all $j \in S \setminus \{i\}$, $\xi(j, S \setminus \{i\}) = \xi(j, S)$, we say i is a* neutral *element of S. We say that ξ is* semi-cross-monotonic*, if every element of every set is either positive, negative, or neutral. In other words, ξ is semi-cross-monotonic if there is no set $S \subseteq \mathscr{A}$ and three distinct elements i, j_1, j_2 of S, such that $\xi(j_1, S \setminus \{i\}) < \xi(j_1, S)$ and $\xi(j_2, S \setminus \{i\}) > \xi(j_2, S)$.*[10]

Thus, cross-monotonicity is precisely a special case of semi-cross-monotonicity, when every element of every set is either positive or neutral. The results in this section are based on the following partial characterization of group-strategyproof mechanisms.

Theorem 4.1 *For every α-budget-balanced group-strategyproof cost-sharing mechanism $\mathcal{M}$ for a cost function C, there is a cost-sharing scheme $\xi_{\mathcal{M}}$ for C such that*

(a) $\xi_{\mathcal{M}}$ is α-budget-balanced and semi-cross-monotonic.

(b) for any set S and bid vector $\mathbf{b}$ such that $b_i = -1$ for $i \notin S$ and $b_i > \xi_{\mathcal{M}}(i, S)$ for $i \in S$, the mechanism $\mathcal{M}$ services the set S.

(c) for any bid vector $\mathbf{b}$, if the serviced set is S, then the payment of $i \in S$ is equal to $\xi_{\mathcal{M}}(i, S)$.

We note that this is not a complete characterization of group-strategyproof mechanisms, as there are semi-cross-monotonic cost-sharing schemes that do not correspond to any group-strategyproof mechanism (See Appendix B). Finding a complete characterization of cost-sharing schemes that give rise to group-strategyproof mechanisms is an interesting open direction.

Before proving the above theorem, we state two of the corollaries of this theorem. These results characterize group-strategyproof mechanisms that satisfy the following additional properties.

Definition 4.2 *A mechanism $\mathcal{M}$ is* upper continuous *if for every agent i, if i gets the service for every bid value greater than x holding other bids fixed, then i gets the service if he bids x.*

[10] Notice that this definition allows sets that contain both negative and positive elements. Also, an element can be a positive element of one set and a negative element of another.

Definition 4.3 *A mechanism is* subsidy-free *if, for any bid vector, the total charge to any subset of agents is at most the cost of servicing that subset.*

Although arguably not well-motivated, the condition of upper-continuity allows us to prove the following equivalence between cross-monotonic cost-sharing schemes and group-strategyproof mechanisms satisfying this condition, hence implying that all the upper bounds on the budget-balance factor of cross-monotonic cost-sharing schemes proved in Section 3 apply to such mechanisms as well. This theorem can be viewed as guidance in the search for group-strategyproof mechanisms: in order to design a mechanism with better revenue properties than the best cross-monotonic cost-sharing schemes, one must build a mechanism which violates upper continuity.

Theorem 4.2 *The cost function C has an upper-continuous α-budget-balanced group-strategyproof mechanism if and only if it has an α-budget-balanced cross-monotonic cost-sharing scheme.*

The subsidy-freeness property was considered previously by Moulin [21]. This property parallels the core condition of cost-sharing games and is motivated by the argument that no subset of serviced agents should be over-charged to accommodate others. The following theorem shows the equivalence of group-strategyproof mechanisms satisfying this property and cross-monotonic cost-sharing schemes, in the case that the mechanism is perfectly budget balanced. We do not know if this theorem holds for budget-balance factors other than 1, and so the results of Section 3 only imply that the problems presented there do not have budget-balanced group-strategyproof mechanisms satisfying subsidy-freeness.

Theorem 4.3 *The cost function C has a subsidy-free budget-balanced group-strategyproof mechanism if and only if it has a budget-balanced cross-monotonic cost-sharing scheme.*

In the rest of this section, we present the proofs of Theorems 4.1, 4.2, and 4.3.

Proof of Theorem 4.1. **(a):** We start by defining the cost-sharing scheme $\xi_{\mathcal{M}}$. For an agent i, let b_i^* be a large enough value such that if agent i bids b_i^*, she will get the service, independent of other agents' bids (such a value exists by CS). For a set $S \subseteq \mathscr{A}$, consider the scenario where the agents in S bid their value in $\mathbf{b}^*$, and others bid -1. By CS and VP, the set of agents serviced by the mechanism in this scenario is precisely S. We define the cost share $\xi_{\mathcal{M}}(i, S)$ as the payment charged by the mechanism to the agent i in this scenario. By this definition and the fact that $\mathcal{M}$ is α-budget balanced, it is clear that $\xi_{\mathcal{M}}$ is also α-budget balanced.

Now, we prove that $\xi_{\mathcal{M}}$ is semi-cross-monotonic. Assume, for contradiction, that there is a set $S \subseteq \mathscr{A}$ and three distinct agents $i, j_1, j_2 \in S$ such that $\xi(j_1, S \setminus \{i\}) < \xi(j_1, S)$ and $\xi(j_2, S \setminus \{i\}) > \xi(j_2, S)$. Consider three bid vectors $\mathbf{b}^1, \mathbf{b}^2$, and $\mathbf{b}^3$ defined as follows: In all of these vectors, agents $j \in S \setminus \{i\}$ bid b_j^* and agents $j \in \mathscr{A} \setminus S$ bid -1. The bid of i in these vectors is $b_i^1 = b_i^*$, $b_i^2 = \xi_{\mathcal{M}}(i, S)$, and $b_i^3 = -1$. By VP and CS, the set of serviced agents at $\mathbf{b}^1$ is S, at $\mathbf{b}^3$ is $S \setminus \{i\}$, and at $\mathbf{b}^2$ is either S or $S \setminus \{i\}$. Furthermore, by the definition of $\xi_{\mathcal{M}}$, the payment of each agent j at the bid vectors $\mathbf{b}^1$ and $\mathbf{b}^3$ is $\xi(j, S)$ and $\xi(j, S \setminus \{i\})$, respectively. We consider two cases based on whether i is serviced at the bid vector $\mathbf{b}^2$:

Case 1: i is served at the bid vector $\mathbf{b}^2$. By VP, i's payment at $\mathbf{b}^2$ is at most $b_i^2 = \xi_{\mathcal{M}}(i, S)$. If i's payment is *strictly* less than $\xi_{\mathcal{M}}(i, S)$, then in a scenario where the utility of the agents is given by $\mathbf{b}^1$, i would have an incentive to announce a bid of b_i^2, contradicting the strategyproofness of the mechanism. Therefore, when all agents bid according to $\mathbf{b}^2$, the payment of i must be equal to $\xi_{\mathcal{M}}(i, S)$. Now consider the payment $x_{j_1}(\mathbf{b}^2)$ of j_1 when agents bid $\mathbf{b}^2$. If $x_{j_1}(\mathbf{b}^2) < \xi(j_1, S)$, then in the scenario where the utility of the agents is given by $\mathbf{b}^1$, $\{i, j_1\}$ can form a successful coalition: they can bid according to $\mathbf{b}^2$, thereby decreasing the payment of j_1, and not changing the payment of i. Also, if $x_{j_1}(\mathbf{b}^2) > \xi(j_1, S \setminus \{i\})$, then in the scenario where the utility of the agents is given by $\mathbf{b}^2$, $\{i, j_1\}$ can form a successful coalition: they can bid according to $\mathbf{b}^1$. This decreases the payment of j_1, and i is indifferent between the two situations, as her utility is zero in both. Thus, $\xi(j_1, S) \le x_{j_1}(\mathbf{b}^2) \le \xi(j_1, S \setminus \{i\})$, contradicting the definition of j_1.

Case 2: i is not served at the bid vector $\mathbf{b}^2$. Consider the payment $x_{j_2}(\mathbf{b}^2)$ of j_2 when agents bid $\mathbf{b}^2$. If $x_{j_2}(\mathbf{b}^2) < \xi(j_2, S \setminus \{i\})$, then if the true utility of the agents is given by $\mathbf{b}^3$, $\{i, j_2\}$ can form a coalition: they can bid according to b^2, thereby reducing j_2's payment while keeping the utility of i constant at zero. Also, if $x_{j_2}(\mathbf{b}^2) > \xi(j_2, S)$, then if the utility of the agents is given by b^2, $\{i, j_2\}$ can form a coalition and bid according to b^1, thereby reducing j_2's payment and keeping i's utility constant at zero. Therefore, $\xi(j_2, S \setminus \{i\}) \le x_{j_2}(\mathbf{b}^2) \le \xi(j_2, S)$, contradicting the definition of j_2.

The contradiction in both cases shows that $\xi_{\mathcal{M}}$ is semi-cross-monotonic.

(b): Index the agents such that $S = \{1, \ldots, k\}$. For $i = 0, \ldots, k$, define the bid vector $\mathbf{b}^{(i)}$ as follows: $b_j^{(i)} = b_j^*$ for $1 \le j \le k - i$, $b_j^{(i)} = b_j > \xi_{\mathcal{M}}(j, S)$ for $k - i < j \le k$, and $b_j^{(i)} = -1$ for $j \in \mathscr{A} \setminus S$. We will prove by induction on i that if the agents bid $\mathbf{b}^{(i)}$, then the mechanism $\mathcal{M}$ will service the agents in S and charges $j \in S$ an amount equal to $\xi_{\mathcal{M}}(j, S)$. This statement for $i = k$ would imply (b). The induction basis ($i = 0$) is obvious from CS and the definition of $\xi_{\mathcal{M}}$. To show the induction step, we assume that the statement is true for i and prove it for $i + 1$. The only difference between the bid vectors $\mathbf{b}^{(i)}$ and $\mathbf{b}^{(i+1)}$ is the bid of the agent $k - i$. If at the bid vector $\mathbf{b}^{(i+1)}$ agent $k - i$ is either not serviced, or is charged an amount more than $\xi_{\mathcal{M}}(k - i, S)$, then this agent has an incentive to announce a bid of b_{k-i}^* when the true utilities of the agents is given by $\mathbf{b}^{(i+1)}$. Similarly, if $k - i$ is serviced and charged an amount less than $\xi_{\mathcal{M}}(k - i, S)$ when agents bid according to $\mathbf{b}^{(i+1)}$, then when the true utilities of the agents is given by $\mathbf{b}^{(i)}$, agent $k - i$ has an incentive to bid b_{k-i}. Therefore, at $\mathbf{b}^{(i+1)}$, $k - i$ gets serviced and pays $\xi_{\mathcal{M}}(k - i, S)$. This means that from the perspective of agent $k - i$, outcomes at $\mathbf{b}^{(i)}$ and $\mathbf{b}^{(i+1)}$ are the same. Therefore, for every other agent j, the agent j must be indifferent between these two outcomes as well, since otherwise $\{i, j\}$ can form a coalition at one of the two bid vectors $\mathbf{b}^{(i)}$ or $\mathbf{b}^{(i+1)}$. Therefore, by the induction hypothesis, at the bid vector $\mathbf{b}^{(i+1)}$, every agent $j \in S$ must receive the service and be charged $\xi(j, S)$.

(c): Let $S_1 = \{i \in S \mid b_i \le \xi_{\mathcal{M}}(i, S)\}$, $S_2 = S \setminus S_1$, and $S_3 = \mathscr{A} \setminus S$. By VP, every $i \in S_1$ is not charged more than $\xi_{\mathcal{M}}(i, S)$ at $\mathbf{b}$. Suppose the price charged to some agent $i^* \in S_1$ is strictly less than $\xi(i^*, S)$. Consider a bid vector $\mathbf{b}'$ in which every agent $i \in S_1$ bids b_i^*, every $i \in S_2$ bids b_i (his bid in $\mathbf{b}$) and every $i \in S_3$ bids -1. From part (b), at the bid vector $\mathbf{b}'$, set S will receive the service and $i \in S$ will pay $\xi_{\mathcal{M}}(i, S)$. Now, since the agent $i^* \in S_1$ is charged strictly less than $\xi_{\mathcal{M}}(i^*, S)$ at $\mathbf{b}$, then when the true utilities are given by $\mathbf{b}'$, i^* can form a coalition with the agents in $S_1 \cup S_3$ and submit the bid vector $\mathbf{b}$. As a result, i^* pays strictly less and no member of the coalition pays more, contradicting group-strategyproofness. Therefore the price of any agent $i \in S_1$ equals $\xi_{\mathcal{M}}(i, S)$ at the bid vector $\mathbf{b}$.

Now consider an agent $i \in S_2$. If his payment differs between $\mathbf{b}$ and $\mathbf{b}'$, then i can form a coalition with the agents in $S_1 \cup S_3$ and submit the bid vector in which he pays less. Agent i strictly benefits from this, while the situation of the agents in $S_1 \cup S_3$ does not change, again contradicting the group-strategyproofness of $\mathcal{M}$. Therefore the payment of every agent $i \in S_2$ also equals $\xi_{\mathcal{M}}(i, S)$. $\square$

Proof of Theorem 4.2. The "if" part of this statement follows from Theorem A and the simple observation that the Moulin mechanism $\mathcal{M}_\xi$ is upper continuous.

Given an α-budget-balanced group-strategyproof mechanism $\mathcal{M}$, we show that the cost-sharing scheme $\xi_{\mathcal{M}}$ defined in the proof of Theorem 4.1 is cross-monotonic. In other words, we need to show that every element of every set is either positive or neutral. Define $\mathbf{b}^*$ as in the proof of Theorem 4.1. Consider a set $S \subseteq \mathscr{A}$ and an agent $i \in S$. Let $\mathbf{b}$ be a bid vector such that $b_j = b_j^*$ for every $j \in S \setminus \{i\}$, $b_j = -1$ for every $j \in \mathscr{A} \setminus S$, and b_i is any number greater than $\xi_{\mathcal{M}}(i, S)$. By part (b) of Theorem 4.1, at any such bid vector, the set S gets the service. Therefore, by the upper continuity of $\mathcal{M}$ and CS, the set S gets the service when i bids $\xi_{\mathcal{M}}(i, S)$ and every other agent bids according to $\mathbf{b}$. Call this bid vector $\mathbf{b}'$.

Now, assume, for contradiction, that $\xi_{\mathcal{M}}(j, S \setminus \{i\}) < \xi_{\mathcal{M}}(j, S)$ for some $j \in S \setminus \{i\}$. We argue that $\{i, j\}$ can form a successful coalition when the utilities of the agents is given by $\mathbf{b}'$. In this situation, if i bids -1 and j does not change her bid, then by Theorem 4.1 the set $S \setminus \{i\}$ receives the service and agent j pays $\xi_{\mathcal{M}}(j, S \setminus \{i\})$. This outcome makes the agent j strictly happier, and agent i is indifferent between the two outcomes. This contradicts the group-strategyproofness of $\mathcal{M}$. This contradiction shows that every element i of every set S is either positive or neutral, and hence $\xi_{\mathcal{M}}$ is cross-monotonic. $\square$

Proof of Theorem 4.3. As in the previous proof, the "if" direction is a direct corollary of Theorem A and the simple observation that $\mathcal{M}_\xi$ satisfies subsidy-freeness.

Given a subsidy-free 1-budget-balanced mechanism $\mathcal{M}$, we show that the cost-sharing scheme $\xi_{\mathcal{M}}$ defined in Theorem 4.1 is cross-monotonic. First, notice that by part (c) of Theorem 4.1, subsidy-freeness of $\mathcal{M}$ implies that $\xi_{\mathcal{M}}$ is in the 1-core of C, that is, for every $T \subseteq S \subseteq \mathscr{A}$, we have

$$\sum_{j \in T} \xi_{\mathcal{M}}(j, S) \leq C(T). \tag{8}$$

Now, consider a set $S \subseteq \mathscr{A}$ and an agent $i \in S$. If i is a negative element of S, then for every $j \in S \setminus \{i\}$, we have $\xi_{\mathcal{M}}(j, S) \geq \xi_{\mathcal{M}}(j, S \setminus \{i\})$, and at least for one j this inequality is strict. Therefore,

$$\sum_{j \in S \setminus \{i\}} \xi_{\mathcal{M}}(j, S) > \sum_{j \in S \setminus \{i\}} \xi_{\mathcal{M}}(j, S \setminus \{i\}) = C(S - \{i\}), \tag{9}$$

where the last equality follows from the fact that $\mathcal{M}$ is 1-budget-balanced. Equation 9 contradicts Equation 8 for $T = S \setminus \{i\}$. $\square$

5 Conclusion

In this paper, we studied upper bounds for the budget-balance factor of cross-monotonic cost-sharing schemes for a variety of combinatorial optimization games. Our techniques are quite general and may prove applicable to a variety of other combinatorial games. For example, Könemann et al. [19] used techniques similar to the ones introduced in this paper to solve an open question posed in the conference version of this paper regarding the Steiner tree game. As another example, the facility location game restricted to a tree always has a budget-balanced cost allocation in the core [10], but we do not have a tight lower and upper bound on the budget-balance factor of the best cross-monotonic cost-sharing schemes for this game. For the facility location game on the line, we have an upper bound of $\frac{6}{7}$.

An interesting open question is to fully characterize cost-sharing schemes that can arise as $\xi_{\mathcal{M}}$ for some group-strategyproof mechanism $\mathcal{M}$. The results of Section 4.3 is a step toward solving this problem. Another open question is to generalize Theorem 4.3 for mechanisms with budget-balance factors less than one.

Finally, we would like to note that there was an error in Example 4.2 and Theorem 4.2 of the conference version of this paper [13]: The mechanism in Example 4.2 can be poorly budget-balanced, and the mechanism in Theorem 4.2 is not group-strategyproof. We would like to thank Hervé Moulin for noticing the latter mistake.

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A A group-strategyproof mechanism with no cross-monotonic cost-sharing scheme

In this appendix, we give an example that shows that for some cost functions, group-strategyproof mechanisms do not correspond to cross-monotonic cost-sharing schemes.

Example A.1 *Suppose there are three agents,* 1, 2, *and* 3, *with a cost function given by*

$$C(S) = \begin{cases} 2 & \textit{if } |S| = 3, \\ 1 & \textit{otherwise}. \end{cases}$$

We consider the following mechanism for this cost function:

Mechanism $\mathcal{M}$:

If $b_1 \geq 1$ *then*
  If $\min(b_2, b_3) > \frac{1}{2}$ *then* $Q = \{1, 2, 3\}$ *and* $x = (1, \frac{1}{2}, \frac{1}{2})$,
  else if $\max(b_2, b_3) < \frac{1}{2}$ *then* $Q = \{1\}$ *and* $x = (1, 0, 0)$,
  else if $b_2 \geq b_3$ *then* $Q = \{1, 2\}$ *and* $x = (\frac{1}{2}, \frac{1}{2}, 0)$,
  else if $b_3 > b_2$ *then* $Q = \{1, 3\}$ *and* $x = (\frac{1}{2}, 0, \frac{1}{2})$,
else if $\frac{1}{2} \leq b_1 < 1$ *then*
  If $\min(b_2, b_3) > \frac{1}{2}$ *then* $Q = \{2, 3\}$ *and* $x = (0, \frac{1}{2}, \frac{1}{2})$,
  else if $\max(b_2, b_3) < \frac{1}{2}$ *then* $Q = \emptyset$ *and* $x = (0, 0, 0)$,

else if $b_2 \geq b_3$ *then* $Q = \{1, 2\}$ *and* $x = (\frac{1}{2}, \frac{1}{2}, 0)$,
else if $b_3 > b_2$ *then* $Q = \{1, 3\}$ *and* $x = (\frac{1}{2}, 0, \frac{1}{2})$,
else if $b_1 < \frac{1}{2}$ *then*
If $\min(b_2, b_3) \geq \frac{1}{2}$ *then* $Q = \{2, 3\}$ *and* $x = (0, \frac{1}{2}, \frac{1}{2})$,
else if $b_2 \geq 1$ *then* $Q = \{2\}$ *and* $x = (0, 1, 0)$,
else if $b_3 \geq 1$ *then* $Q = \{3\}$ *and* $x = (0, 0, 1)$,
else $Q = \emptyset$ *and* $x = (0, 0, 0)$.

The cost-sharing scheme $\xi_{\mathcal{M}}$ is not cross-monotonic since, for example, $\xi_{\mathcal{M}}(1, \{1, 2, 3\}) > \xi_{\mathcal{M}}(1, \{1, 2\})$. In fact, it is not hard to see that no cross-monotonic cost-sharing scheme for C exists. Still, as the following lemma shows, the mechanism $\mathcal{M}$ is group-strategyproof.

Proposition A.1 *The mechanism $\mathcal{M}$ in Example A.1 is group-strategyproof.*

Proof. Let u_i denote the true utility of i for receiving the service, b_i denote his bid, and $x_i(\mathbf{b})$ denote his payment when the bids are $\mathbf{b}$. Note $x_i(\mathbf{b}) = 0$ if and only if i does not receive the service.

We first prove by contradiction that any successful coalition must include 1. Suppose not (that is, $b_1 = u_1$). First consider the case $u_1 \geq \frac{1}{2}$. Note that for $i \in \{2, 3\}$, whenever i receives the service, he pays $\frac{1}{2}$. Therefore, i can benefit only if $u_i > \frac{1}{2}$ and he is not receiving service. However, in any input bid vector with $b_1 \geq \frac{1}{2}$, $b_i > \frac{1}{2}$ implies that i receives the service, so i can not benefit in any coalition. Next suppose $u_1 < \frac{1}{2}$. Consider the cross-monotonic cost-sharing scheme $\xi : \{2, 3\} \to \mathbb{R}_+$ where for $i \in \{2, 3\}$, $\xi(i, \{2, 3\}) = \frac{1}{2}$ and $\xi(i, \{i\}) = 1$. The Moulin mechanism $\mathcal{M}_\xi$ is equivalent to $\mathcal{M}$ when $u_1 < \frac{1}{2}$ and so Theorem A implies that there is no subset of $\{2, 3\}$ can form a successful coalition in this case.

Now consider any coalition including 1. Suppose $u_1 < \frac{1}{2}$. If $b_1 < \frac{1}{2}$, then the outcome does not change if we set $b_1 = u_1$. Thus, we only need to consider coalitions in which $b_1 \geq \frac{1}{2}$. As $u_1 < \frac{1}{2}$ and the minimum non-zero price of 1 is $\frac{1}{2}$, it must be that $1 \notin Q(\mathbf{b})$ even though $b_1 \geq \frac{1}{2}$. This happens only when $\frac{1}{2} \leq b_1 < 1$ and $\max(b_2, b_3) < \frac{1}{2}$ or $\min(b_2, b_3) > \frac{1}{2}$. In the first case, as no agent receives service, all utilities are zero and so no one can benefit. In the second case, for $i \in \{2, 3\}$, the payment of i is $\frac{1}{2}$. Therefore, if i is in the coalition, it must be that $u_i \geq \frac{1}{2}$. If i is not in the coalition, then $u_i = b_i > \frac{1}{2}$ by assumption. Thus $\min(u_2, u_3) \geq \frac{1}{2}$. But then $x(\mathbf{b}) = x(\mathbf{u})$ and so no agent's utility for the outcome changes.

Next, suppose $u_1 \geq \frac{1}{2}$. For $i \in \{2, 3\}$, in the truthful scenario i pays at most $\frac{1}{2}$. As i's payment is always at least $\frac{1}{2}$, i can not benefit from a decrease in price. Therefore i can benefit only if $u_i > \frac{1}{2}$ and $i \notin Q(\mathbf{u})$. But this is impossible for any vector with $u_1 \geq \frac{1}{2}$, so i can not benefit in any coalition. Therefore, 1 must be the agent that benefits from the coalition. As the minimum price for 1 is $\frac{1}{2}$, in order for 1 to benefit, it must be that $u_1 > \frac{1}{2}$ but either $1 \notin Q(\mathbf{u})$ or $x_1(\mathbf{u}) = 1$. This means that either $\min(u_2, u_3) > \frac{1}{2}$ (case one) or $\max(u_2, u_3) < \frac{1}{2}$ (case two). Furthermore, 1 can only benefit if $x_1(\mathbf{b}) = \frac{1}{2}$ since, when $u_1 \geq 1$, 1 is receiving the service at price 1 and so the price must decrease, and when $\frac{1}{2} \leq u_1 < 1$, 1 is not receiving the service but can not afford to pay 1 and so must receive the service at price $\frac{1}{2}$. Now, in case one ($\min(u_2, u_3) > \frac{1}{2}$), in the truthful scenario 2 and 3 have positive utility. In order for $x_1(\mathbf{b}) = \frac{1}{2}$, i for $i = 2$ or $i = 3$ must lower his bid to $b_i \leq \frac{1}{2}$. But then if the coalition consists of just i and 1, $i \notin Q(\mathbf{b})$ and so i's utility decreases. Similarly, if the coalition is $\{1, 2, 3\}$, then 1 only benefits if $\{2, 3\} \not\subset Q(\mathbf{b})$ and so the utility of 2 or 3 decreases. In case two ($\max(u_2, u_3) < \frac{1}{2}$), 1 can only benefit if i for $i = 2$ or $i = 3$ raises his bid to $b_i \geq \frac{1}{2}$. But then if the coalition consists of just i and 1, $x_i(\mathbf{b}) = \frac{1}{2}$ and so i's utility becomes negative. Similarly, if the coalition is $\{1, 2, 3\}$, then at least one of 2 or 3 must pay $\frac{1}{2}$, and so his utility becomes negative. $\square$

B A semi-cross-monotonic cost-sharing scheme with no group-strategyproof mechanism

Suppose there are just two agents, 1 and 2. The cost of servicing both agents is 6 while the cost of servicing either agent individually is 1. The following is a budget-balanced semi-cross-monotonic cost-sharing scheme:

$$\xi(1, \{1, 2\}) = \xi(2, \{1, 2\}) = 3, \quad \xi(1, \{1\}) = \xi(2, \{2\}) = 1$$

However, this scheme can not correspond to the payments in any group-strategyproof mechanism. First consider the bid vector $\mathbf{b}^1 = (3, 3)$. By group-strategyproofness, the mechanism must service exactly one of the agents; otherwise they could collude and bid either $(-1, 2)$ or $(2, -1)$. Without loss of generality, suppose it services agent 2. Now consider the bid vector $\mathbf{b}^2 = (3, 2)$. Again, the mechanism must service agent 2 since otherwise he could bid 3 and get the service at price 1. Finally, consider the bid vector $\mathbf{b}^3 = (b_1^*, 2)$, where b_1^* is as in the proof of Theorem 4.1. Now the mechanism must service just agent 1 at price 1. But this implies that in bid vector $\mathbf{b}^2$, agent 1 could have profitably deviated by bidding b_1^*.

Remark B.1 *In this cost-sharing scheme, removing either agent from the set* $\{1, 2\}$ *decreased the cost share of the other agent. This property allowed us to draw conclusions about the serviced set in bid vector* $\mathbf{b}^1$ *which led us to our contradiction. This highlights the following general fact: if two agents* i *and* j *are both negative in a set* S*, then either* $\xi(i, S \setminus \{j\}) = \xi(i, S)$ *or* $\xi(j, S \setminus \{i\}) = \xi(j, S)$ *(or both).*