Market Design: Lecture 2

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Recap

3. b) Structure, polytopes: integrality of stable matching polytope, applications
4. a) Incentives: dominant strategies

Outline

4. b) Incentives: complete information Nash equilibria, incomplete information
5. Many-to-one markets: responsive preferences, substitutable preferences

Part 4: Incentives.

M-Optimal Mechanisms

Theorem [Dubins-Freedman ’81, Roth ‘82]. In any M-optimal mechanism, truth-telling is a dominant strategy for men.

In fact, it is group-strategyproof for men.

Blocking Lemma

Blocking Lemma. Let μ be any IR matching with respect to preferences P and let M' be all men that prefer μ to μ^M. Then if M' is non-empty, there is a blocking pair (m, w) for μ such that m is in M-M' and w is in μ(M').

Blocking Lemma

Given M' prefer μ to μ^M, find blocking pair (m, w) for μ with m in M – M' and w in μ(M').

- $\mu(M') \neq \mu^M(M')$: let w be in $\mu(M') - \mu^M(M')$
  - suppose $w = \mu(m')$ and note m' in μ(M') so $w >_{m'} \mu^M(m')$
  - since μ^M stable, $m = \mu^M(w) >_w m'$
  - m is not in M' since w is not in μ^M(M')
  - hence $w >_m \mu(m)$

Blocking Lemma

- $\mu(M') = \mu^M(M') = W'$: let w be last woman in W' to receive a proposal from acceptable man in M' in DA
  - all women in W' have rejected acceptable offers from men in M' (namely their match in μ)
  - thus w is engaged with some m when she gets proposal
  - note m is not in M' since otherwise he'd propose to someone else in W' contradicting assumption
  - also note by property of DA, $w >_m \mu^M(m)$, and since m is not in M', $\mu^M(m) >_m \mu(m)$, so $w >_m \mu(m)$
  - since w has previously rejected $\mu(w)$, must be $m >_w \mu(w)$

Coalition-Proofness

Theorem. Let P be the true preferences and P' differ from P in that some coalition C of men and women misstate their preferences. Then there is no matching μ, stable for P', which is preferred to *every* stable matching under P by all members of C.

(So men-proposing DA group-strategyproof for men.)

Coalition-Proofness

Prf. Suppose M' U W' benefit by reporting P'.

- resulting matching μ is IR under true prefs P
- clearly, μ not stable under P and
  - $\mu(m) >_m \mu^M(m)$ for all m in M'
  - $\mu(w) >_w \mu^W(w)$ for all w in W'
- say M' non-empty and apply blocking lemma to get (m, w) who both prefer μ^M to μ
- note m not in M', so P'(m) = P(m) (similarly w) and so (m, w) also block under altered prefs

Nash Equilibria

All stable matchings are outcomes at a NE.

Theorem. Let μ be stable for (M, W, P) and construct Q in which men report truthfully and each woman reports $\mu(w)$ as her preference list. Then Q is a NE of the men-proposing DA alg.

Stable μ are Outcome of NE

Prf. Note outcome of Q is μ.

- By previous results, men can't manipulate.
- Suppose w has deviation Q'_w matching her to $m = \mu'(w) >_w \mu(w)$. Will find blocking pair for μ'.
- Consider $w^* = \mu(m)$ and note that both $w^* >_m w$ and $m >_{w^*} w^*$(as μ is stable).
- Since under Q' only acceptable man for w* is m, w* is single in μ'.
- Therefore, (m, w*) block μ' w.r.t. Q'

Nash Equilibria

All equilibrium outcomes are stable.

Theorem. Suppose in reported prefs Q each man states his true preferences and the women play an equilibrium of men-proposing DA. Then corresponding matching μ is stable w.r.t true preferences.

Outcome of NE are Stable μ

Prf. Say μ blocked by (m, w) under true prefs P.

- Since $w >_m \mu(w)$ in P and hence Q, m must have proposed to w and been rejected.
- Consider deviation Q'_w in which w lists m first and then the remaining prefs as in Q.
- Then DA runs same until m proposes to w, at which point she accepts and remains with him.
- Thus Q'_w profitable for w, so Q not equilibrium.

Incomplete Information

- Types drawn according to common prior
- Agents submit ordinal preferences
- Almost exclusively negative results
  - No mech whose equilibria are always stable
  - M-opt stable mech can be manipulated by coalitions of men

… because of distributions over matchings

Incomplete Information

$$P(m_1) = \begin{cases} \text{a. } w_1, w_2 & \text{prob. } 1-q \\ \text{b. } w_1 & \text{prob. } q \end{cases} \qquad P(w_1) = \begin{cases} \text{a. } m_2, m_1 & \text{prob. } 1-q \\ \text{b. } m_2 & \text{prob. } q \end{cases}$$

$P(m_2) =$ w_2, w_1 $\qquad$ $P(w_2) =$ m_1, m_2

Potential stable matchings:

- $\mu = \{(m_1, w_1), (m_2, w_2)\}$,
- $\nu = \{(m_1, w_2), (m_2, w_1)\}$,
- $\lambda = \{(m_1), (m_2, w_2), (w_1)\}$

Mech(P^{aa}) selects, say, μ w/prob. > ½

Stable(P^{aa})= $\{\mu, \nu\}$, Stable(P^{ab})= $\{\nu\}$, Stable(P^{ba})= $\{\mu\}$, Stable(P^{bb})= $\{\lambda\}$

Incomplete Information

$P(m_1) = \begin{cases} \text{a. } w_1, w_2 & \text{prob. } 1\text{-}q \\ \text{b. } w_1 & \text{prob. } q \end{cases}$ $P(w_1) = \begin{cases} \text{a. } m_2, m_1 & \text{prob. } 1\text{-}q \\ \text{b. } m_2 & \text{prob. } q \end{cases}$

$P(m_2) =$ w_2, w_1 $P(w_2) =$ m_1, m_2

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- $\mu = \{(m_1, w_1), (m_2, w_2)\}$,
- $\nu = \{(m_1, w_2), (m_2, w_1)\}$,
- $\lambda = \{(m_1), (m_2, w_2), (w_1)\}$

Mech(P^{aa}) selects, say, μ w/prob. > ½

In state P^{aa}, w_2 profits by reporting $P'(w_2) = m_1$. This occurs with probability $(1\text{-}q)^2$, so beneficial for q small enough.

Incomplete Information

- By revelation principle, no mechanism implements stable outcomes.
- Some large market results

Question. Is there some restricted set of priors for which we can implement stable outcomes?

Part 5: Many-to-One Markets.

Many-to-One Markets

- Colleges $C = \{c_1, \ldots, c_n\}$
- Quotas $q = \{q_1, \ldots, q_n\}$
- Students $S = \{s_1, \ldots, s_p\}$
- Preferences
  - $P(s_i)$ ordered list of colleges
  - $P^*(c_i)$ ordered list of *subsets* of students

College Preferences

How are colleges' preferences over sets related to preferences over individual students?

- pref over students: $P(c) = s_1, s_2, s_3, s_4$
- pref over subsets:
  - $P^*(c) = \{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}, \{s_2, s_3\}, \{s_2, s_4\}, \{s_3, s_4\}$
  - $P^*(c) = \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_2, s_4\}, \{s_3, s_4\}$
  - $P^*(c) \neq \{s_1, s_3\}, \{s_1, s_2\}, \{s_2, s_3\}, \{s_1, s_4\}, \{s_2, s_4\}, \{s_3, s_4\}$

College Preferences

Responsive prefs: for any subset T of students with $|T| < q_c$ and students s, s' not in T,

- $T \cup \{s\} >_c T \cup \{s'\}$ if and only if $s >_c s'$
- $T \cup \{s\} >_c T$ if and only if s is acceptable to c

Potentially many preferences are responsive to same individual ordering.

Stable Matchings

- matching μ maps S U C to S U C s.t.
  - μ(s) is an element of C U {s}
  - μ(c) is a subset of S U {c} of cardinality q_c
  - μ(s) = c if and only if s is in μ(c)
- individually rational if assignments acceptable
- blocked by a pair (c, s) if μ(s) ≠ c and
  - $c >_s \mu(c)$ and $s >_c t$ for some t in μ(c)
- pairwise stable if IR and unblocked by pairs

Group Stability

- matching μ blocked by coalition A of S U C if exists another matching ν such that for all s, c
  - ν(s) is in A
  - $\nu(s) >_s \mu(s)$
  - t in ν(c) implies t in A or t in μ(c)
  - $\nu(c) >_c \mu(c)$
- group stable if not blocked by any coalition

Group vs Pairwise Stability

Lemma. When preferences are responsive, a matching μ is group stable iff it's pairwise stable.

Prf. Pairwise instability implies group instability.

- Suppose μ blocked by A, ν
- Consider s, c such that s is in {ν(c) – μ(c)}
- Responsiveness implies that for some such s there exists a t in {μ(c) – ν(c)} s.t. $s >_c t$
- (s, c) block μ

Group vs Pairwise Stability

Lemma. When preferences are responsive, a matching μ is group stable iff it's pairwise stable.

Implications: if preferences are responsive,

- don't need prefs over sets to find stable matchings
- set of stable matchings not sensitive to prefs over sets (so long as responsive to same P)

Existence

Related one-to-one market:

- q_c copies of each college c each with pref P(c)
- for each s, update P(s) to replace c with its copies in top-down order (strict prefs)

Lemma. Matching stable iff corresponding matching in related one-to-one market stable.

So stable matchings exist (and “DA” works).

What Results Carry Over?

- Optimal stable matchings, lattice structure:
  - S-optimal matchings exist
  - C-optimal?
    Not even clear c can compare μ and ν!
- Incentives:
  - in student-proposing DA, truth-telling is a dominant strategy for students
  - college-proposing DA?
    Colleges create "coalitions."

Optimality

Theorem. College-proposing DA in related one-to-one market gives each college c top q_c achievable students.

College-optimal stable matchings exist.

Optimality

However, C-optimal need not be Pareto optimal:

$P(s_1) = c_3, c_1, c_2$

$P(s_2) = c_2, c_1, c_3$

$P(s_3) = c_1, c_3, c_2$

$P(s_4) = c_1, c_2, c_3$

$P(c_1) = s_1, s_2, s_3, s_4$

$P(c_2) = s_1, s_2, s_3, s_4$

$P(c_3) = s_3, s_1, s_2, s_4$

$q_1 = 2, q_2 = q_3 = 1$

- stable matching μ gives c_1 students s_3, s_4.
- matching $\mu' = \{(c_1, s_2, s_4), (c_2, s_1), (c_3, s_3)\}$ strictly preferred by every college to μ.

Incentives

Theorem. There is no stable matching mech. in which truthtelling is dominant strat. for colleges.

$P(s_1) = c_3, c_1, c_2$

$P(s_2) = c_2, c_1, c_3$

$P(s_3) = c_1, c_3, c_2$

$P(s_4) = c_1, c_2, c_3$

$P(c_1) = s_1, s_2, s_3, s_4$

$P(c_2) = s_1, s_2, s_3, s_4$

$P(c_3) = s_3, s_1, s_2, s_4$

$q_1 = 2, q_2 = q_3 = 1$

- only stable $\mu = \{(c_1, s_3, s_4), (c_2, s_2), (c_3, s_4)\}$
- if c_1 states $P'(c_1) = s_1, s_4, c_1$, then
  only stable $\nu = \{(c_1, s_1, s_4), (c_2, s_2), (c_3, s_3)\}$

Equilibria

No dominant strat for colleges makes it hard to select among NE for college-proposing, but…

Theorem. Even for student-proposing DA when students use dominant strategies, there are equilibria which are not stable w.r.t. true prefs.

In fact, any IR μ is an equilibrium outcome.

Comparing Matchings

Can colleges compare stable matchings?

Example: students ranked by score on exam.

- for any stable matchings μ and ν, no two entering classes have same average score
- in fact, if $\mu(c) >_c \nu(c)$, then for *every* pair of s in μ(c)-ν(c), t in ν(c), score of s ≥ score of t!

Comparing Matchings

Theorem. For college c and any stable μ and ν, if there are students s in $\mu(c^i)$, t in $\nu(c^i)$ s.t. $s >_c t$, then for all j, $\mu(c^j) \geq_c \nu(c^j)$.

Prf. Assume position c^i of college c, $\mu(c^i) >_c \nu(c^i)$ and show for all $j > i$, $\mu(c^j) >_c \nu(c^j)$.

Implications

- If c indifferent between μ and ν, then $\mu(c)=\nu(c)$
- If $\mu(c) = \{s_1, s_4\}$, $\nu(c) = \{s_2, s_3\}$, not both stable
- If $\mu(c) = \{s_1, s_3\}$, $\nu(c) = \{s_1, s_4\}$, not both stable

Also implies existence of lattice structure, etc.

Rural Hospitals

Corollary. If college has vacant positions in some stable matching, then it gets *the same set of students* in every stable matching.

Prf. Recall if c^i of college c, $\mu(c^i) >_c \nu(c^i)$ then for all $j > i$, $\mu(c^j) >_c \nu(c^j)$.

- positions filled in order of preference
- so for vacant positions (high j), $\mu(c^j) =_c \nu(c^j)$
- but if there's a difference in set of students, then $\mu(c^i) >_c \nu(c^i)$ or $\nu(c^i) >_c \mu(c^i)$ for some i