Market Design: Lecture 2

Nicole Immorlica, Northwestern University

Recap

1. Introduction:
   – entry-level labor markets, school choice, kidney exchange
2. Stable Matching Model:
   – Matching μ stable for market (M, W, P) if IR and no blocking pairs (m, w) where $w >_m \mu(m)$ and $m >_w \mu(w)$.
   – Stable matchings exist and can be computed via DA.
   – Men-proposing DA computes M-optimal (W-pessimal) stable matching μ^M.

Recap

3. a) Structure, lattices:
   - Stable matchings partially ordered by "pointing function" $\vee_M$ is a complete distributive lattice.
   - Every finite complete distributive lattice equals the set of stable matchings for some preferences.

Outline

3. b) Structure, polytopes: integrality of stable matching polytope, applications
4. Incentives: dominant strategies, complete information Nash equilibria, incomplete information
5. Many-to-one markets: responsive preferences, substitutable preferences

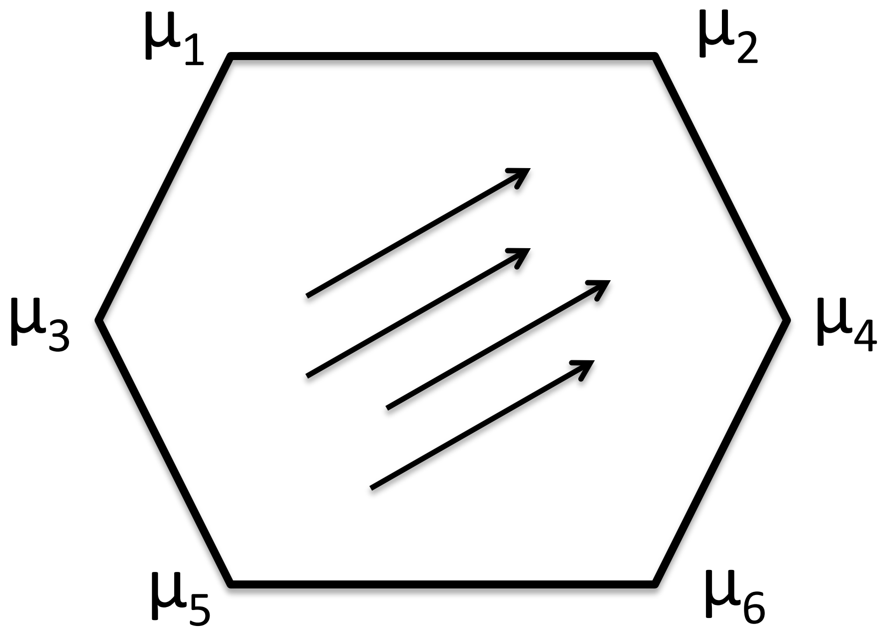
Part 3b: Structure, polytopes.

Stable Matching Polytope

- Market (M, W, P) with complete preferences
- Variables $x_{mw} = 1$ if $\mu(m) = w$, 0 otherwise
- Matching constraints:
  - for all m, $\Sigma_{j \text{ in } W} x_{mj} = 1$
  - for all w, $\Sigma_{i \text{ in } M} x_{iw} = 1$
  - for all m, w, $x_{mw} \geq 0$
- Stability constraint:
  - for all m, w, $\Sigma_{j >_m w} x_{mj} + \Sigma_{i >_w m} x_{iw} + x_{mw} \geq 1$

Stable Matching Polytope

Theorem [Vande Vate '89]. Stable matching polytope is the convex hull of stable matchings.



Stable Matching Polytope

matching polytope, integral by Birkhoff

$$\begin{cases} \Sigma_{j \text{ in } W}\, x_{mj} = 1 \\ \Sigma_{i \text{ in } M}\, x_{iw} = 1 \\ x_{mw} \geq 0 \end{cases}$$

$$\Sigma_{j >_m w}\, x_{mj} + \Sigma_{i >_w m}\, x_{iw} + x_{mw} \geq 1$$

Stable Matching Polytope

Integrality proof [Sethuraman-Teo '98].

Goal: show any feasible fractional $\{x_{mw}\}$ can be written as convex combination of integral solns.

Idea: rounding scheme with expectation $\{x_{mw}\}$.

Stable Matching Polytope

Lemma. Tight constraints:

$x_{mw} > 0$ implies $\Sigma_{j >_m w}\, x_{mj} + \Sigma_{i >_w m}\, x_{iw} + x_{mw} = 1$.

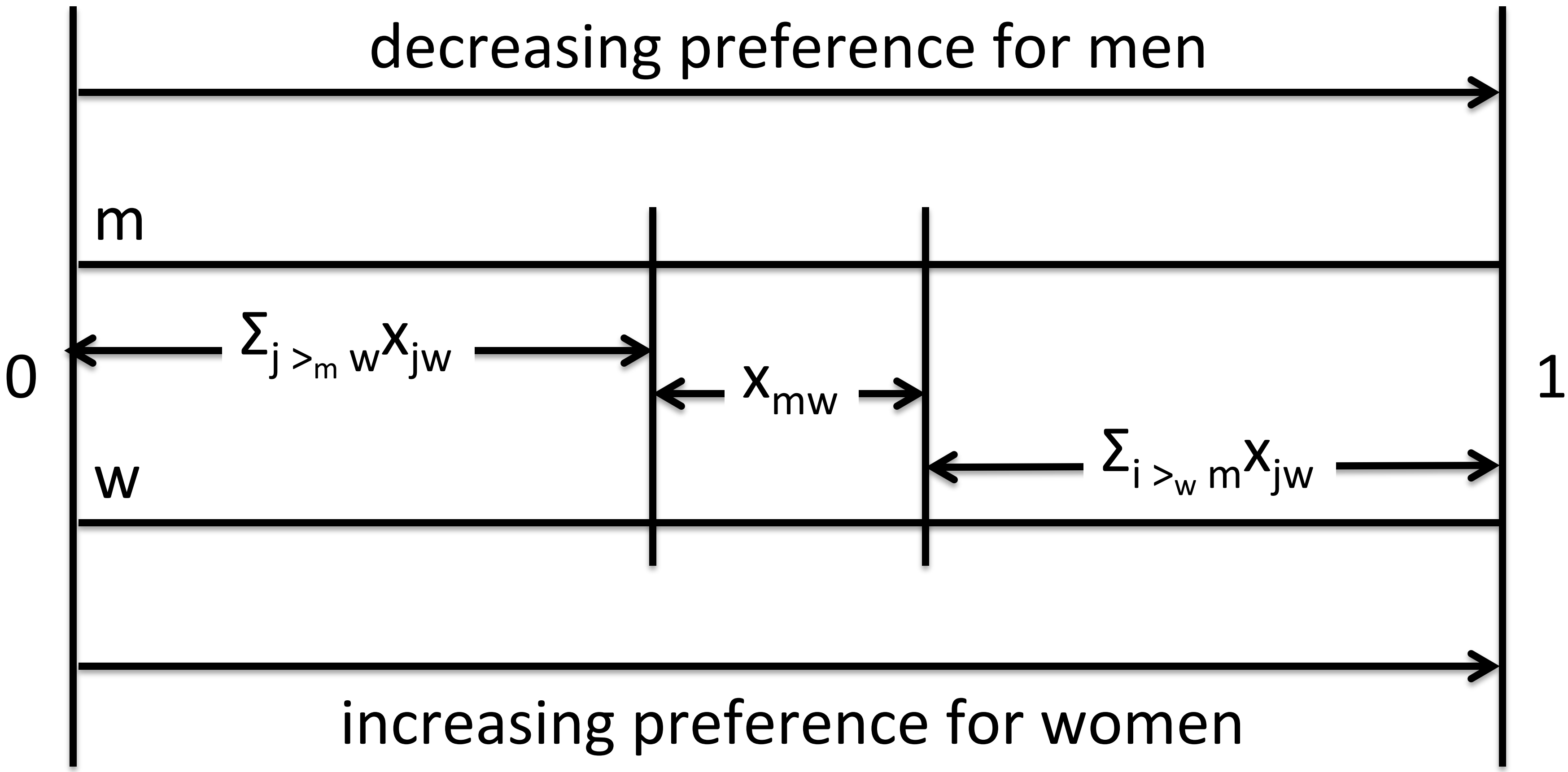
Prf. Complementary slackness.

Stable Matching Polytope

$x_{mw} > 0$ implies $\Sigma_{j >_m w}\, x_{mj} + \Sigma_{i >_w m}\, x_{iw} + x_{mw} = 1$

- Create table of 2n unit intervals
  - row m: place x_{mj} in decreasing order to cover [0,1]
  - row w: place x_{iw} in increasing order to cover [0,1]
- Tightness implies intervals line up!

Stable Matching Polytope



$x_{mv} > 0$ implies $\Sigma_{j >_m w}\, x_{mj} + \Sigma_{i >_w m}\, x_{iw} + x_{mw} = 1$

Stable Matching Polytope

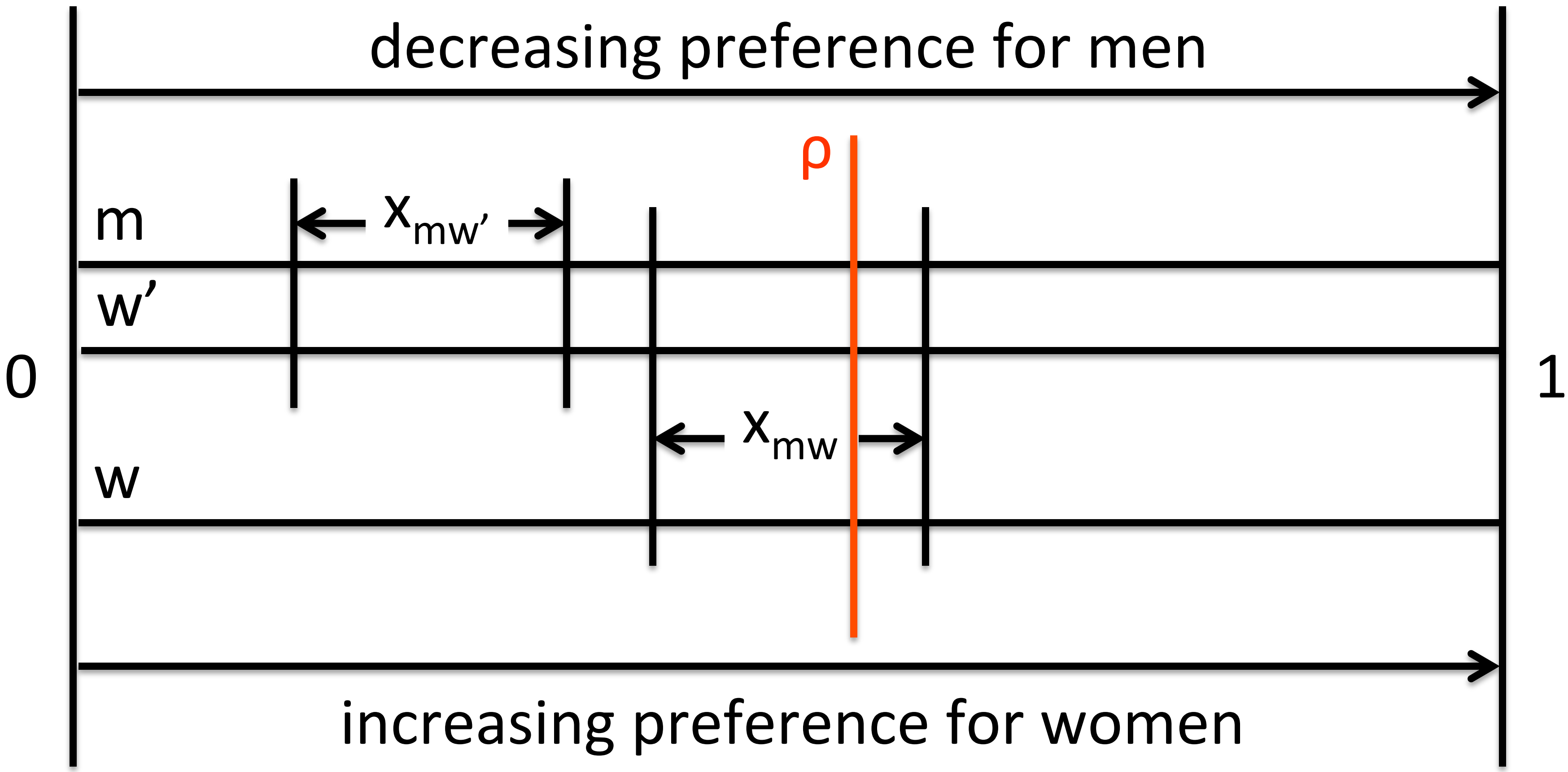
Rounding Scheme:

- Choose random ρ in [0,1]
- Mate of m is w if ρ in sub-interval of (m,w)

Observations:

- Since intervals line up, defines a matching
- Since preferences anti-aligned, stable

Stable Matching Polytope



Stable Matching Polytope

- rounding gives stable integral matching x^ρ
- $E[x^\rho_{mw}] = Pr[\rho \text{ in sub-interval } (m,w)] = x_{mw}$
- therefore, fractional soln convex comb of integral ones

Question. Applications to computational questions (finding all achievable pairs, etc.)

Median Matchings

Theorem [Sethuraman-Teo]. Consider matchings $\mu_1, \ldots, \mu_l$ and note that each person has l mates (with repetition).

- assign each man his k'th ranked woman
- assign each woman her (l-k+1)'th ranked man

Then resulting assignment is a stable matching.

Median Matchings

Prf. Let x^i be variables for matching μ_i.

- Define $x_{mw} = (1/l)\ \Sigma_i\ x^i_{mw}$.
- Consider $\rho = k/l - \varepsilon$ for arbitrarily small ε.
- Rounding with ρ gives promised assignment so stable matching by previous argument.

* Can also be proved directly using lattice result.

Median Matchings

Corollary. For l odd, k = (l+1)/2, shows there exists a "median" stable matching.

Question. Can we efficiently compute the median stable matching?

Recommended Reading

- Sethuraman, Teo, and Qian. Many-to-One Stable Matching: Geometry and Fairness, *Mathematics of Operations Research*, 2006.

Part 4: Incentives.

Stable Marriage Game

- marriage mechanism: maps reported preferences to matchings
- strategy-proof if reporting true preferences is a dominant strategy for the agents

Example: DA Not Strategy-Proof

$P(m_1) = w_1, w_2$ $P(w_1) = m_2, m_1$

$P(m_2) = w_2, w_1$ $P(w_2) = m_1, m_2$

- men-proposing deferred-acceptance gives
  $\mu^M = \{(m_1, w_1), (m_2, w_2)\}$
- if w_2 reports $P'(w_2) = m_1$, then mech. gives
  $\mu^W = \{(m_1, w_2), (m_2, w_1)\}$

Questions

- Is there any strategy-proof matching mechanism that yields stable outcomes?
- In DA, what are the incentives? Which agents want to lie, and how and when?
- How do these manipulations on the part of one agent affect the welfare of the others?

Strategy-Proof Mechanisms

1. There exist strategy-proof matching mechs

2. No stable mechanism is strategy-proof

Strategy-Proof Mechanisms

Example: Pareto optimal matching

- order men $m_1, \ldots, m_n$
- for i = 1 to n, match m_i to top choice among remaining women

Mechanism is strategy-proof, Pareto optimal, but *not stable*.

Strategy-Proof Mechanisms

Pareto optimal mechanism in practice:

- professional sports teams drafting college players, order "men" (teams) worst to best
- US Naval Academy procedure for assigning students to posts, order "men" (students) best to worst

Aside: used in practice and not stable, why?

Impossibility Theorem

Theorem [Roth]. No stable strategy-proof mech.

Prf. Consider following example:

$P(m_1) = w_1, w_2$ $\qquad$ $P(w_1) = m_2, m_1$

$P(m_2) = w_2, w_1$ $\qquad$ $P(w_2) = m_1, m_2$

- Only stable solns are μ^M, μ^W
- Say mech. outputs μ^M with prob. > ½
- Then w_2 can deviate by $P'(w_2) = m_1$ in which case only stable soln is μ^W

Impossibility Theorem

Theorem [Roth]. No stable strategy-proof mech.

Corollary. No truth-telling Nash equilibrium.

In fact, *whenever* there is more than one stable matching, at least one agent has incentive to misreport when others are truthful.

Rural Hospitals

Theorem. The set of men and women that are unmatched is same at every stable matching.

Remark. Explains why rural hospitals are undersubscribed no matter what.

Rural Hospitals

Prf. For matching μ, let

- $\mu(M)$ be set of matched women,
- $\mu(W)$ be set of matched men.

Consider M-optimal μ^M and arbitrary μ.

$$\left|\mu^M(W)\right| \supseteq \left|\mu(W)\right|$$

$$\| \qquad\qquad \|$$

$$\left|\mu^M(M)\right| \subseteq \left|\mu(M)\right|$$

μ^M is best for men

(# matched men equals # matched women at any μ)

μ^M is worst for women

Rural Hospitals

Prf. For matching μ, let

- $\mu(M)$ be set of matched women,
- $\mu(W)$ be set of matched men.

Consider M-optimal μ^M and arbitrary μ.

- $\mu^M(W)$ subset of $\mu(W)$ and $|\mu^M(W)| = |\mu(W)|$
- so $\mu^M(W) = \mu(W)$.

Impossibility Theorem

Theorem. If there is more than one stable matching, at least one agent has incentive to misreport when others are truthful.

Impossibility Theorem

Prf. Suppose ≥ 2 stable matchings.

- By assumption, $\mu^M \neq \mu^W$
- Suppose mech picks μ^W with prob. < 1 and $\mu \neq \mu^W$ with prob. > 0
- For w s.t. $\mu(w) \neq \mu^W(w)$, set $P'(w) = \mu^W(w)$
- Then μ^W still stable for reported preferences
- Hence w matched at all stable matchings
- So w must be matched to $\mu^W(w) >_w \mu(w)$

M-Optimal Mechanisms

Recall: in deferred-acceptance, men get their optimal achievable mates.

Question. Do men have an incentive to lie?

M-Optimal Mechanisms

Theorem [Dubins-Freedman ’81, Roth ‘82]. In any M-optimal mechanism, truth-telling is a dominant strategy for men.

In fact, it is group-strategyproof for men.

Blocking Lemma

Blocking Lemma. Let μ be any IR matching with respect to preferences P and let M' be all men that prefer μ to μ^M. Then if M' is non-empty, there is a blocking pair (m, w) for μ such that m is in M-M' and w is in μ(M').

Blocking Lemma

Given M' prefer μ to μ^M, find blocking pair (m, w) for μ with m in M – M' and w in μ(M').

- $\mu(M') \neq \mu^M(M')$: let w be in $\mu(M') - \mu^M(M')$
  - suppose $w = \mu(m')$ and note m' in μ(M') so $w >_{m'} \mu^M(m')$
  - since μ^M stable, $m = \mu^M(w) >_w m'$
  - m is not in M' since w is not in μ^M(M')
  - hence $w >_m \mu(m)$

Blocking Lemma

- $\mu(M') = \mu^M(M') = W'$: let w be last woman in W' to receive a proposal from acceptable man in M' in DA
  - all women in W' have rejected acceptable offers from men in M' (namely their match in μ)
  - thus w is engaged with some m when she gets proposal
  - note m is not in M' since otherwise he'd propose to someone else in W' contradicting assumption
  - also note by property of DA, $w >_m \mu^M(m)$, and since m is not in M', $\mu^M(m) >_m \mu(m)$, so $w >_m \mu(m)$
  - since w has previously rejected μ (w), must be $m >_w \mu(w)$

Coalition-Proofness

Theorem. Let P be the true preferences and P' differ from P in that some coalition C of men and women misstate their preferences. Then there is no matching μ, stable for P', which is preferred to *every* stable matching under P by all members of C.

(So men-proposing DA group-strategyproof for men.)

Coalition-Proofness

Prf. Suppose M’ U W’ benefit by reporting P’.

- resulting matching μ is IR under true prefs P
- clearly, μ not stable under P and
  - $\mu(m) >_m \mu^M(m)$ for all m in M’
  - $\mu(w) >_w \mu^W(w)$ for all w in W’
- say M’ non-empty and apply blocking lemma to get (m, w) who both prefer μ^M to μ
- note m not in M’, so P’(m) = P(m) (similarly w) and so (m, w) also block under altered prefs