Market Design: Lecture 1

Nicole Immorlica, Northwestern University

Outline

1. Introduction: two-sided matching markets in practice and impact of theory
2. Stable Matching Model: elementary definitions, fundamental existence result
3. Structure: combinatorial structure of the set of stable matchings, applications

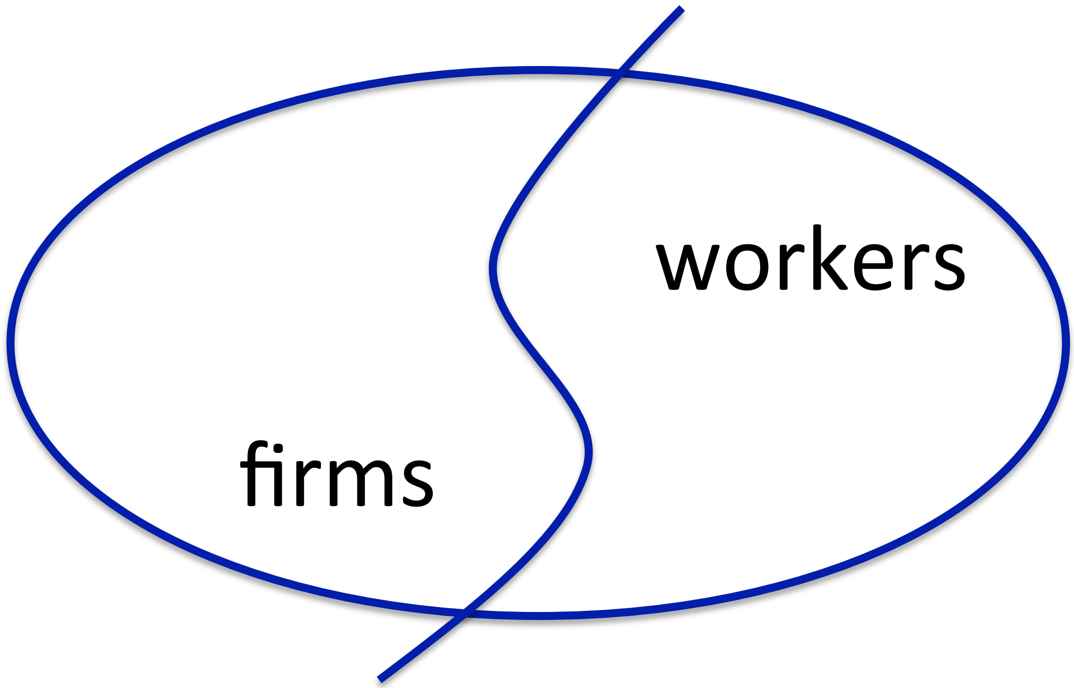
Part 1: Introduction.

Market Design Goal

Develop simple theory,

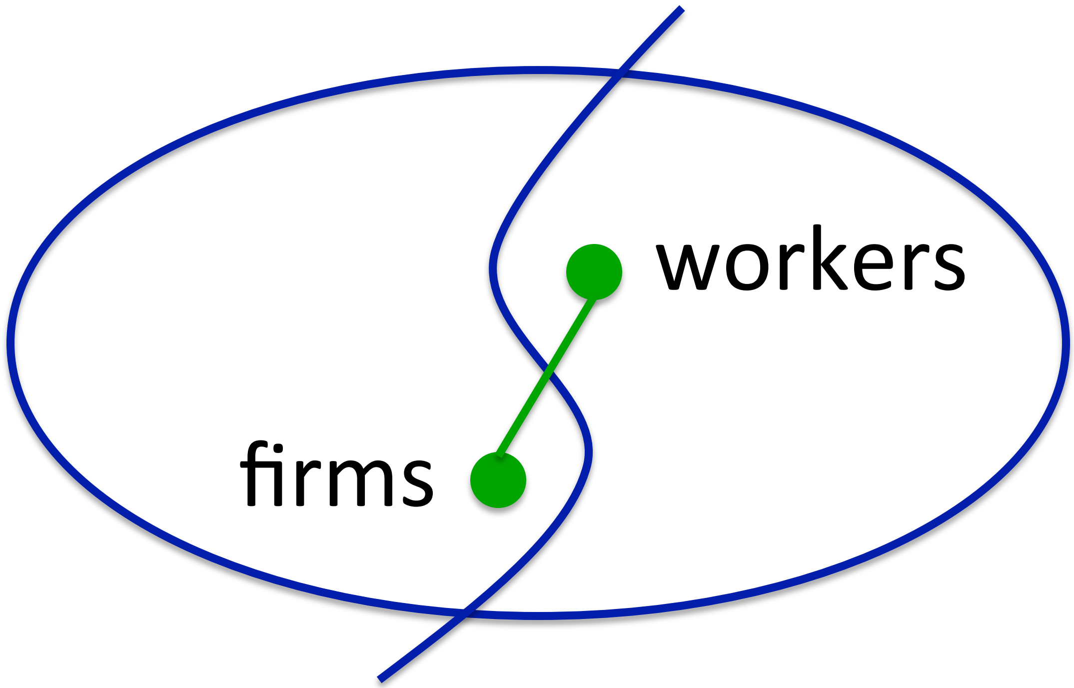
… to deal with complexity in practice

Two-Sided Matching



1. agents partitioned into two disjoint sets
   (as opposed to commodities markets where an agent can be both a buyer and a seller)

Two-Sided Matching



2. bilateral nature of exchange
   (vs. commodities markets where agent sells wheat and buy tractors although wheat-buyer doesn't sell tractors and tractor-seller doesn't buy wheat)

Example: entry-level labor markets

Practice:

National Residency Matching Program (NRMP): physicians look for residency programs at hospitals in the United States

Example: entry-level labor markets

Practice:

1950	1990	
decentralized, unraveling, inefficiencies	centralized clearinghouse, 95% voluntary participation	dropping participation sparks redesign to accommodate couples, system still in use

Example: entry-level labor markets

Theory:

Gale-Shapley stable marriage algorithm:
NRMP central clearinghouse algorithm corresponds to GS algorithm, and so evolution of market resulted in "correct" mechanism

Example: entry-level labor markets

Issues:

- Understanding agents' incentives
- Distribution of interns to rural hospitals
- Dealing with couples
- Preference formation

Example: school choice

Practice:

Boston, New York City, etc:
students submit preferences about different schools; matched based on "priorities" (e.g., test scores, geography, sibling matches)

Example: school choice

Practice:

some mechanisms strategically complicated, result in unstable matches, many complaints in school boards

Example: school choice

Theory:

theorists proposed alternate mechanisms including top-trading cycles and GS algorithm for stable marriage, schools adopt these

Example: school choice

Issues:

- Fairness and affirmative action goals
- Respecting improvements in schools

Example: kidney exchange

Practice:

In 2005:

- 75,000 patients waiting for transplants
- 16,370 transplants performed (9,800 from deceased donors, 6,570 from living donors)
- 4,200 patients died while waiting

Example: kidney exchange

Practice:

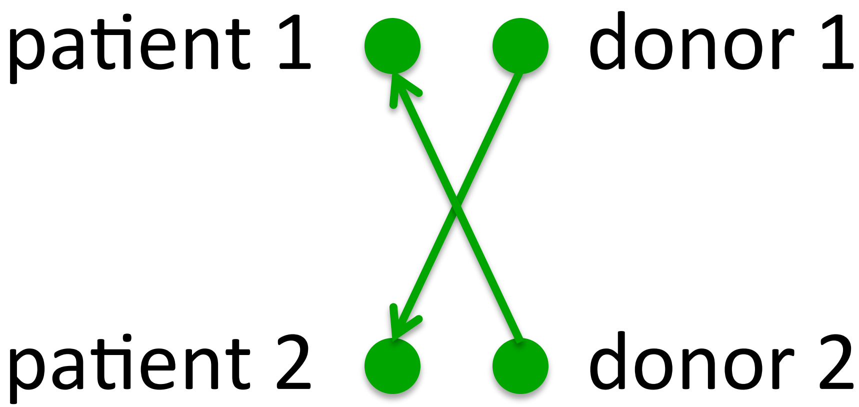
Source and allocation of kidneys:

- cadaver kidneys: centralized matching mechanism based on priority queue
- living donors: patient must identify donor, needs to be compatible
- other: angel donors, black market sales

Example: kidney exchange

Theory:

living donor exchanges:



Example: kidney exchange

Theory:

living donor exchanges:
adopted mechanism uses top-trading cycles, theory of maximum matching, results in improved welfare (many more transplants)

Example: kidney exchange

Issues:

- Larger cycles of exchanges
- Hospitals' incentives

Part 2: Stable Matching Model.

Stable Matching Model

- men $M = \{m_1, \ldots, m_n\}$
- women $W = \{w_1, \ldots, w_p\}$
- preferences
  - $a >_x b$ if x prefers a to b
  - $a \geq_x b$ if x is likes a at least as well as b

Stable Matching Model

- preference lists
  - P(m) ordered list of W U {m}
    $P(m) = w_1, w_2, m, w_3, \ldots, w_p$
    (m prefers being single to marrying $w_3, \ldots, w_p$)
  - P(w) ordered list of M U {w}
    $P(w) = m_1, [m_2, m_3], m_4, \ldots, m_n, w$
    (w is indifferent between m_2 and m_3)

Stable Matching Model

- preferences
  - *strict* if no indifferences
    (we assume strict unless otherwise stated)
  - *rational* by assumption
    (preferences transitive and form a total ordering)
- matchings μ
  - a correspondence μ from set M U W onto itself
    s.t. if μ(m) ≠ m then μ(m) in W (and vice versa)

Stable Matching Model

- matchings
  - $\mu(m)$ is *mate* of m
  - $\mu >_x \nu$ if $\mu(x) >_x \nu(x)$
    (no externalities: m cares only about own mate)

Stable Matching Model

- matching μ is *stable* if
  - individually rational, unblocked by individuals:

    $\mu(x) \geq_x x$ for all x

    (agents can choose to be single, so IR only if every agent is acceptable to mate)
  - unblocked by pairs:

    if $y >_x \mu(x)$, then $\mu(y) >_y x$ for all x,y

    (no pairwise deviations from matching)
- aside: stable iff in core (no subset deviations)

Example

$P(m_1) = w_2, w_1, w_3$

$P(m_2) = w_1, w_3, w_2$

$P(m_3) = w_1, w_2, w_3$

$P(w_1) = m_1, m_3, m_2$

$P(w_2) = m_3, m_1, m_2$

$P(w_3) = m_1, m_3, m_2$

- all matchings are individually rational (since all pairs mutually acceptable)
- $\mu = \{(w_1, m_1), (w_2, m_2), (w_3, m_3)\}$ unstable (since blocked by (m_1, w_2))
- $\nu = \{(w_1, m_1), (w_2, m_3), (w_3, m_2)\}$ is stable

Prediction

"Only stable matchings will occur."

- complete information and easy access
(else blocking pairs persist because agents don't know about each other or can't find each other)
- good idea when participation is voluntary
- many theorems require strict preferences
(indifference unlikely because "knife-edge")

Non-Existence

One-sided (roommate problem):

- n single people to be matched in pairs
- each person ranks n-1 others
- matching stable if no blocking pairs
- example: people {a, b, c, d}

  P(a) = b, c, d P(c) = a, b, d

  P(b) = c, a, d P(d) = arbitrary

  no stable matching since person with d blocks

Other Non-existence

Three-sided (man-woman-child):

- Preferences over pairs of other agents
- (m, w, c) block μ if
  $(w, c) >_m \mu(m)$; $(m, c) >_w \mu(w)$; $(m, w) >_c \mu(c)$

One-to-many (workers-firms):

- Firms have preferences over sets of workers
- Firm f and subset of workers C block μ if
  $C >_f \mu(f)$ and for all w in C, $f >_w \mu(w)$

Existence

First attempt: rejection chains

- start with arbitrary matching
- Repeat until no blocking pairs
  - take arbitrary blocking pair (m,w)
  - match m and w
  - declare mates of m and w to be single

Example: rejection chains

$P(m_1) = w_2, w_1, w_3$

$P(w_1) = m_1, m_3, m_2$

$P(m_2) = w_1, w_3, w_2$

$P(w_2) = m_3, m_1, m_2$

$P(m_3) = w_1, w_2, w_3$

$P(w_3) = m_1, m_3, m_2$

- $\mu_1 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$ blocked by (m_1, w_2)
- $\mu_2 = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$ blocked by (m_3, w_2)
- $\mu_3 = \{(m_1, w_3), (m_2, w_1), (m_3, w_2)\}$ blocked by (m_3, w_1)
- $\mu_4 = \{(m_1, w_3), (m_2, w_2), (m_3, w_1)\}$ blocked by (m_1, w_1)
- $\mu_1 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$ blocked by (m_1, w_2)

Example: rejection chains

$P(m_1) = w_2, w_1, w_3$

$P(m_2) = w_1, w_3, w_2$

$P(m_3) = w_1, w_2, w_3$

$P(w_1) = m_1, m_3, m_2$

$P(w_2) = m_3, m_1, m_2$

$P(w_3) = m_1, m_3, m_2$

Note:

- $\{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$ also blocked by (m_3, w_2)
- $\{(m_1, w_1), (m_2, w_3), (m_3, w_2)\}$ is stable

There are always chains that lead to stable matching!

Rejection chains

Theorem [Roth-Vande Vate '90]. For any matching μ, there exists a finite sequence of matchings μ_1, …, μ_k such that

- $\mu = \mu_1$,
- μ_k is stable, and
- for each i = 1, …, k-1, there is a blocking pair (m,w) for μ_i s.t. μ_{i+1} is obtained from μ_i by matching (m,w) and making their mates single

Rejection chains

Prf. Take arbitrary μ and subset S of agents s.t.

- S does not contain any blocking pairs for μ
- add arbitrary agent x to S
  - if x blocks μ with an agent in S, chain at (x,y) where y is most preferred mate of x among those in S that form a blocking pair with x
  - repeat until no agents in S block μ
- continue growing S until all agents in S

Rejection chains

Prf. Take arbitrary μ and subset S of agents s.t.

- S does not contain any blocking pairs for μ
- add arbitrary agent x to S

deferred acceptance algorithm with respect to S
(terminates with no blocking pairs in S, see next section)

- continue growing S until all agents in S

Rejection chains

Corollary. Random chains converge to stable matching with probability one.

Question. Rate of convergence?

Question. Same results in more general settings (e.g., many-to-many matchings)?

Existence: deferred acceptance

Initiate:

- Each man proposes to 1st choice.
- Each woman rejects all but most preferred acceptable proposal.

Repeat (until no more rejections):

- Any man rejected at previous step proposes to most preferred woman that has not yet rejected him (if such a woman exists).
- Each woman rejects all but most preferred acceptable proposal.

Existence: deferred acceptance

Theorem [Gale-Shapley ‘62]. A stable matching exists for any marriage market.

Prf. The deferred acceptance algorithm computes a stable matching.

Existence: deferred acceptance

Prf. (of men-proposing)

- Terminates: finite number of women, each man proposes to each woman at most once.
- Stable:
  - suppose (m, w) not matched and $w >_m \mu(w)$
  - then m proposed to w and was rejected
  - w must have rejected m for a preferred m'
  - as w's options improve, $\mu(w) \geq_w m' >_w m$
  - so (m, w) not a blocking pair

Example: men-proposing

$P(m_1) = w_2, w_1, w_3$

$P(m_2) = w_1, w_3, w_2$

$P(m_3) = w_1, w_2, w_3$

$P(w_1) = m_1, m_3, m_2$

$P(w_2) = m_3, m_1, m_2$

$P(w_3) = m_1, m_3, m_2$

1. Proposals: $\{(m_1, w_2), (m_2, w_1), (m_3, w_1)\}$
   Intermediate μ: $\{(m_1, w_2), (m_2), (w_3), (m_3, w_1)\}$
2. Proposals: $\{(m_2, w_3)\}$
   Final μ: $\{(m_1, w_2), (m_2, w_3), (m_3, w_1)\}$

Example: women-proposing

$P(m_1) = w_2, w_1, w_3$

$P(m_2) = w_1, w_3, w_2$

$P(m_3) = w_1, w_2, w_3$

$P(w_1) = m_1, m_3, m_2$

$P(w_2) = m_3, m_1, m_2$

$P(w_3) = m_1, m_3, m_2$

1. Proposals: $\{(m_1, w_1), (m_3, w_2), (m_1, w_3)\}$
   Intermediate μ: $\{(m_1, w_1), (m_3, w_2), (m_2), (w_3)\}$
2. Proposals: $\{(m_3, w_3)\}$
   Intermediate μ: $\{(m_1, w_1), (m_3, w_2), (m_2), (w_3)\}$
3. Proposals: $\{(m_2, w_3)\}$
   Final μ: $\{(m_1, w_1), (m_3, w_2), (m_2, w_3)\}$

Properties

$P(m_1) = w_2, w_1, w_3$ $\qquad$ $P(w_1) = m_1, m_3, m_2$

$P(m_2) = w_1, w_3, w_2$ $\qquad$ $P(w_2) = m_3, m_1, m_2$

$P(m_3) = w_1, w_2, w_3$ $\qquad$ $P(w_3) = m_1, m_3, m_2$

$\mu^M = \{(m_1, w_2), (m_2, w_3), (m_3, w_1)\}$

$\mu^W = \{(m_1, w_1), (m_2, w_3), (m_3, w_2)\}$

Each man prefers μ^M to μ^W;
each woman prefers μ^W to μ^M!

Why Not Disagree

$P(m_1) = w_1, w_2, w_3$

$P(m_2) = w_1, w_3, w_2$

$P(m_3) = w_1, w_2, w_3$

$P(w_1) = m_1, m_2, m_3$

$P(w_2) = m_1, m_2, m_3$

$P(w_3) = m_1, m_3, m_2$

- Among *all* matchings, each man likes a different one best (i.e., one where he gets w_1)
- Two *stable* matchings:
  - $\mu = \{(m_1,w_1), (m_2,w_2), (m_3,w_3)\}$
  - $\nu = \{(m_1,w_1), (m_2,w_2), (m_3,w_3)\}$

} stability eliminates disagreement

Common Interests

- Define $\mu \geq_M \nu$ if
  - for all men m, $\mu(m) \geq_m \nu(m)$,
  - $\mu >_M \nu$ if also for some m, $\mu(m) >_m \nu(m)$
  - note this is a partial order and transitive
- μ is M-optimal if, for all stable ν, $\mu \geq_M \nu$
- Similarly, define $\mu \geq_W \nu$ and W-optimal

Common Interests

Theorem [Gale-Shapley '62]. There is always a unique M-optimal stable matching. The matching μ^M produced by the men-proposing deferred acceptance algorithm is M-optimal (similarly for women).

Common Interests

Prf. Define m and w to be *achievable* for each other if matched in some stable matching.

Claim. No man is rejected by an achievable woman.

- By induction: assume until step k, no man is rejected by an achievable woman.
- At k+1, suppose m proposes to w and is rejected.
- If m is unacceptable to w ($m >_m w$), we are done.

Common Interests

Claim. No man is rejected by an achievable woman.

- Else w rejects m in favor of some m’, so $m' >_w m$.
- Note m’ prefers w to all women except those who previously rejected him (who are unachievable by inductive hypothesis).
- Suppose w achievable for m and let μ be stable matching that matches them.
- Then μ(m’) achievable for m and μ(m’) ≠ w.
- So $w >_{m'} \mu(m')$ and thus (m’, w) block μ.

Opposing Interests

Men and women disagree.

Theorem [Knuth '76]. For stable matchings μ and ν, $\mu >_M \nu$ if and only if $\nu >_W \mu$.

Corollary. M-optimal matching is worst stable matching for women (each woman gets least-preferred achievable mate) and vice versa.

Opposing Interests

Prf.

Suppose $\mu >_M \nu$ and for some w, $\mu(w) >_w \nu(w)$.

- Then $m = \mu(w)$ has a different mate ν.
- Thus, by assumption, $w = \mu(m) >_m \nu(w)$.
- Thus, (m, w) block matching ν, contradiction.

Part 3: Structure.

Pointing Phenomenon

- Point to your most-preferred mate
  - two men may point to same woman
- Point to your most-preferred *achievable* mate
  - each man points to a different woman!
  - resulting matching is stable!
- What about pointing among mates in arbitrary stable matchings μ and ν?
  - men point to different woman, matching stable

Pointing Phenomenon

- Define $\lambda = \mu \vee_M \nu$ as
  - assign each man more-preferred mate: $\lambda(m) = \mu(m)$ if $\mu(m) >_m \nu(m)$; else $\lambda(m) = \nu(m)$.
  - assign each woman less-preferred mate: $\lambda(w) = \mu(w)$ if $\nu(w) >_w \mu(w)$; else $\lambda(w) = \nu(w)$.
- Is λ a stable matching?
  - if $\lambda(m) = \lambda(m')$, does $m = m'$?
  - if $\lambda(m) = w$, does $\lambda(w) = m$?
  - is λ stable?

Pointing Phenomenon

Theorem [Conway]. If μ and ν are stable, then $\lambda = \mu \vee_M \nu$ is a stable matching (also $\mu \wedge_M \nu$).

Prf. First show λ is a matching, i.e., $\lambda(m) = w$ if and only if $\lambda(w) = m$.

- if $\lambda(m) = w$ then $w = \mu(m) >_m \nu(m)$, so stability of ν requires $\nu(w) >_w \mu(w) = m$ implying $\lambda(w) = m$.
- if $\lambda(w) = m$, must worry about unmatched case…

Pointing Phenomenon

Prf. Next show λ is stable.

- suppose (m, w) block λ
- then $w >_m \lambda(m)$, so $w >_m \mu(m)$ and $w >_m \nu(m)$
- furthermore, $m >_w \lambda(w)$, so
  - (m, w) block μ if $\lambda(w) = \mu(w)$
  - (m, w) block ν if $\lambda(w) = \nu(w)$

Example

$P(m_1) = w_1, w_2, w_3, w_4$ $P(w_1) = m_4, m_3, m_2, m_1$

$P(m_2) = w_2, w_1, w_4, w_3$ $P(w_2) = m_3, m_4, m_1, m_2$

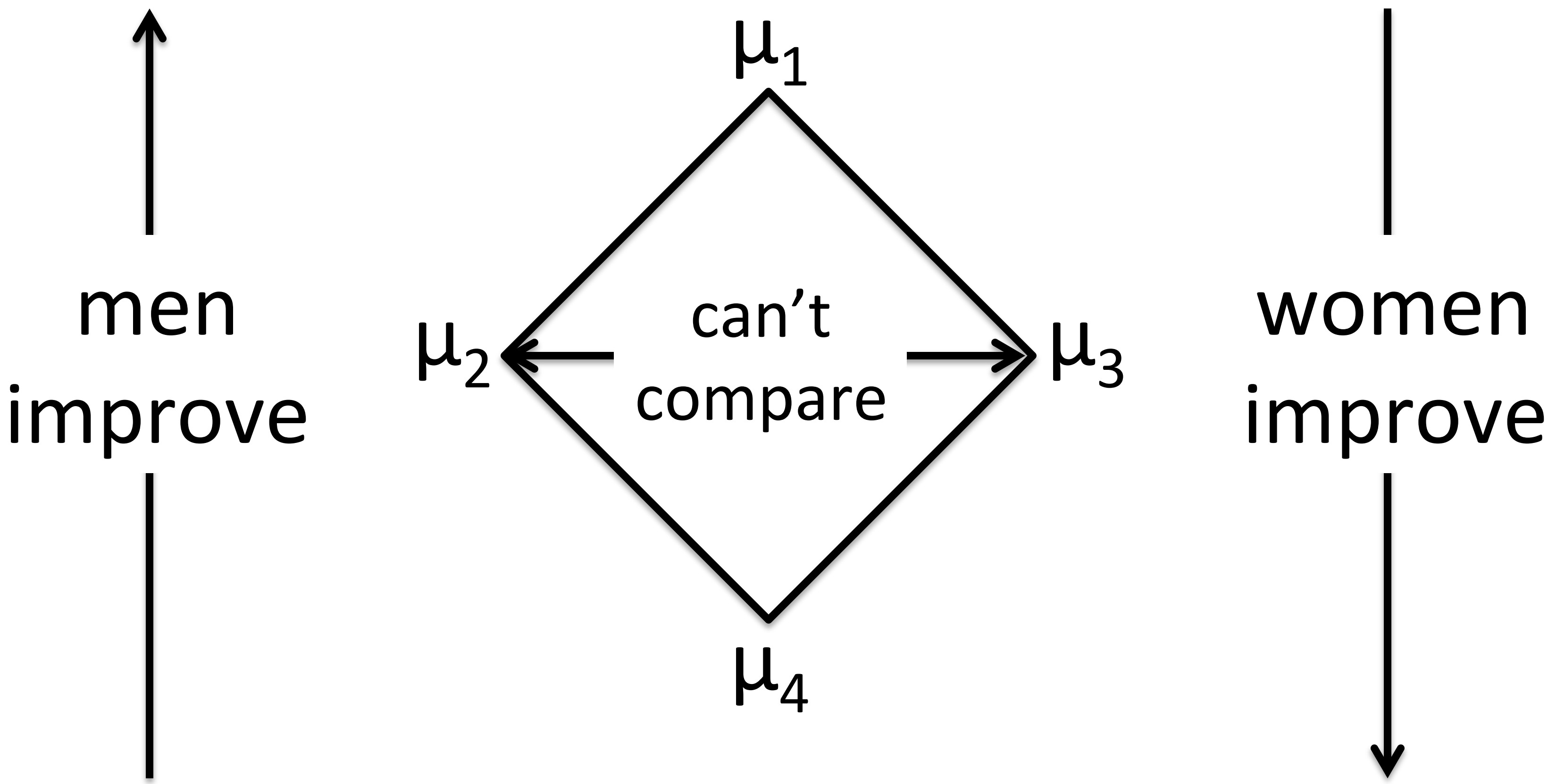
$P(m_3) = w_3, w_4, w_1, w_2$ $P(w_3) = m_2, m_1, m_4, m_3$

$P(m_4) = w_4, w_3, w_2, w_1$ $P(w_4) = m_1, m_2, m_3, m_4$

Ten stable matchings, e.g.,

- $\mu_1 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$
- $\mu_2 = \{(m_1, w_2), (m_2, w_1), (m_3, w_3), (m_4, w_4)\}$
- $\mu_3 = \{(m_1, w_1), (m_2, w_2), (m_3, w_4), (m_4, w_3)\}$
- $\mu_4 = \{(m_1, w_2), (m_2, w_1), (m_3, w_4), (m_4, w_3)\}$

Example



Lattice Structure

Defn. A lattice is a partially ordered set (poset) where every two elts have a least upper bound (join) and greatest lower bound (meet).

… complete if every subset has meet/join.

… distributive if meet/join have distributive law.

Lattice Structure

… lattice is poset where all pairs have meet/join

… complete if every subset has meet/join.

… distributive if meet/join have distributive law.

Eg. S is subset of integers ordered by divisibility:

S	lattice?	complete?	distributive?
{1, 2, 3}	✗	✗	✗
{1, 2, 3, …}	✓	✗	✓
{0, 1, 2, 3, …}	✓	✓	✓

Lattice Structure

- The set of stable matchings partially ordered by "pointing function" $\vee_M$ is a complete distributive lattice, and

- Every finite complete distributive lattice equals the set of stable matchings for some preferences.

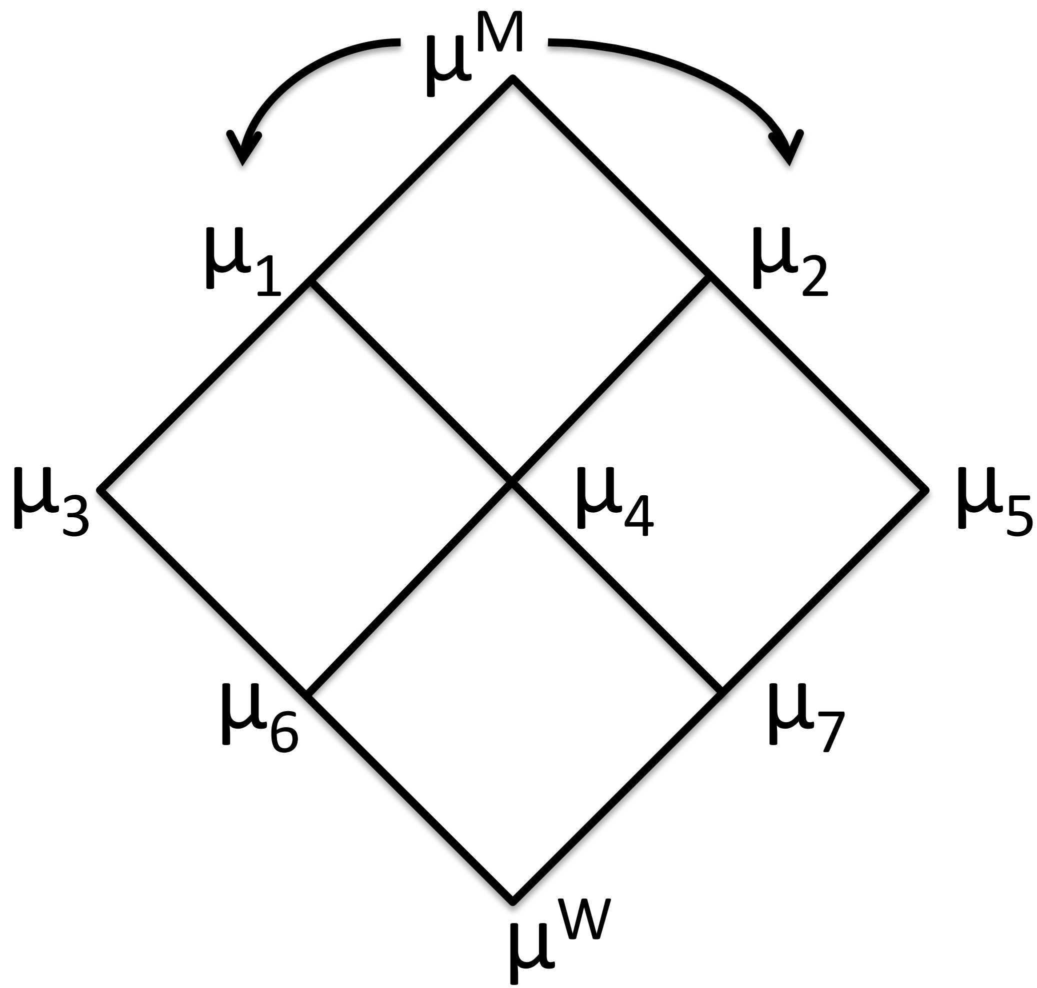
Computational Questions

- Generating all stable matchings
- The number of stable matchings
- Finding all achievable pairs

Generating All Matchings

Idea:

- Compute μ^M
- walk through lattice.



Generating All Matchings

$P(m_1) = w_2, w_1,$ ~~w_3~~

$P(m_2) =$ ~~w_1~~$, w_3,$ ~~w_2~~

$P(m_3) = w_1, w_2,$ ~~w_3~~

$P(w_1) = m_1, m_3,$ ~~m_2~~

$P(w_2) = m_3, m_1,$ ~~m_2~~

$P(w_3) =$ ~~m_1~~, ~~m_3~~, m_2

$\mu^M = \{(m_1, w_2), (m_2, w_3), (m_3, w_1)\}$

$\mu^W = \{(m_1, w_1), (m_2, w_3), (m_3, w_2)\}$

Generating All Matchings

$P(m_1) = w_2, w_1$

$P(m_2) = w_3$

$P(m_3) = w_1, w_2$

$P(w_1) = m_1, m_3$

$P(w_2) = m_3, m_1$

$P(w_3) = m_2$

- First element of P(.) is M-optimal mate
- Last element of P(.) is W-optimal mate
- Can rotate from μ^M to μ^W

Generating All Matchings

$P(m_1) = w_2, w_1$

$P(m_2) = w_3$

$P(m_3) = w_1, w_2$

$P(w_1) = m_1, m_3$

$P(w_2) = m_3, m_1$

$P(w_3) = m_2$

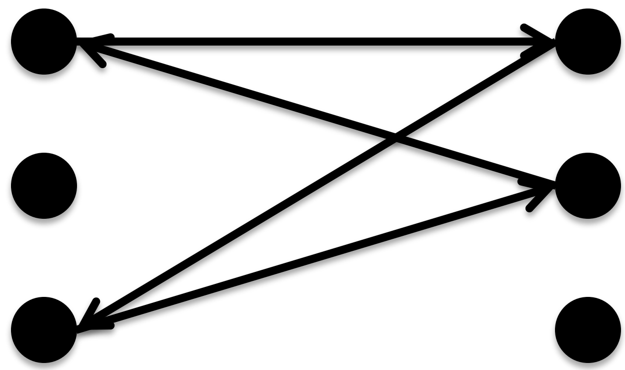
- First element of P(m) is M-optimal mate
- Last element of P(m) is W-optimal mate
- First element of P(w) is W-optimal mate
- Last element of P(w) is M-optimal mate

Generating All Matchings

$P(m_1) = w_2, w_1$

$P(m_2) = w_3$

$P(m_3) = w_1, w_2$



$P(w_1) = m_1, m_3$

$P(w_2) = m_3, m_1$

$P(w_3) = m_2$

- Can rotate from μ^M to μ^W
- Each man points to 2nd favorite woman
- Each woman points to last man
- Perform rotation along cycle

Generating All Matchings

From point μ in lattice, to generate children:

1. Reduce preferences by eliminating women better than μ and worse than μ^W from men, men better than μ^W and worse than μ from women, and all unacceptable people
2. Generate graph according to 2nd-best women for men and worst men $\mu(w)$ for women
3. Perform rotation along each cycle

Polynomial in *number* of stable matchings.

Number of Matchings

- Let n = |M| = |W| and suppose |P(.)| = n.
- Number of *matchings* can be n!

Claim [Irving and Leather '86]. Number of *stable matchings* can be 2^{n-1}.

Number of Matchings

- Double market
  - (M, W, P) with $M = \{m_1,\ldots,m_n\}$ and $W = \{w_1,\ldots,w_n\}$
  - create (M',W',P') with $M'=\{m_{n+1},\ldots,m_{n+n}\}$, $W = \{w_{n+1},\ldots,w_{n+n}\}$, and $P'(x_{n+i}) = P(x_i)_{+n}$
- Merge market
  - m_i and m_{n+i} are partners, w_i and w_{n+i} are partners
  - for men, append partner's preferences to own
  - for women, prepend partner's preferences to own

Number of Matchings

- Given μ stable for (M, W, P) and μ’ stable for (M’, W’, P’),
- set ν(m) = μ(m) if m in M, μ’(m) if m in M’
- set λ(w) = μ(w) if w in W’, μ’(w) if w in M.
- Then ν and λ are stable
- so if (M, W, P) has g(n) stable matchings, then merged market has $2[g(n)]^2$ and size 2n.

Number of Matchings

Claim [Irving and Leather '86]. Number of *stable matchings* can be 2^{n-1}.

Prf. Apply merging to market of size 1.

$$g(1) = 1,\ g(n) = 2[g(n/2)]^2$$

Result follows by solving recurrence.

Number of Matchings

$P(m_1) = w_1, w_2, w_3, w_4$ $\quad$ $P(w_1) = m_4, m_3, m_2, m_1$

$P(m_2) = w_2, w_1, w_4, w_3$ $\quad$ $P(w_2) = m_3, m_4, m_1, m_2$

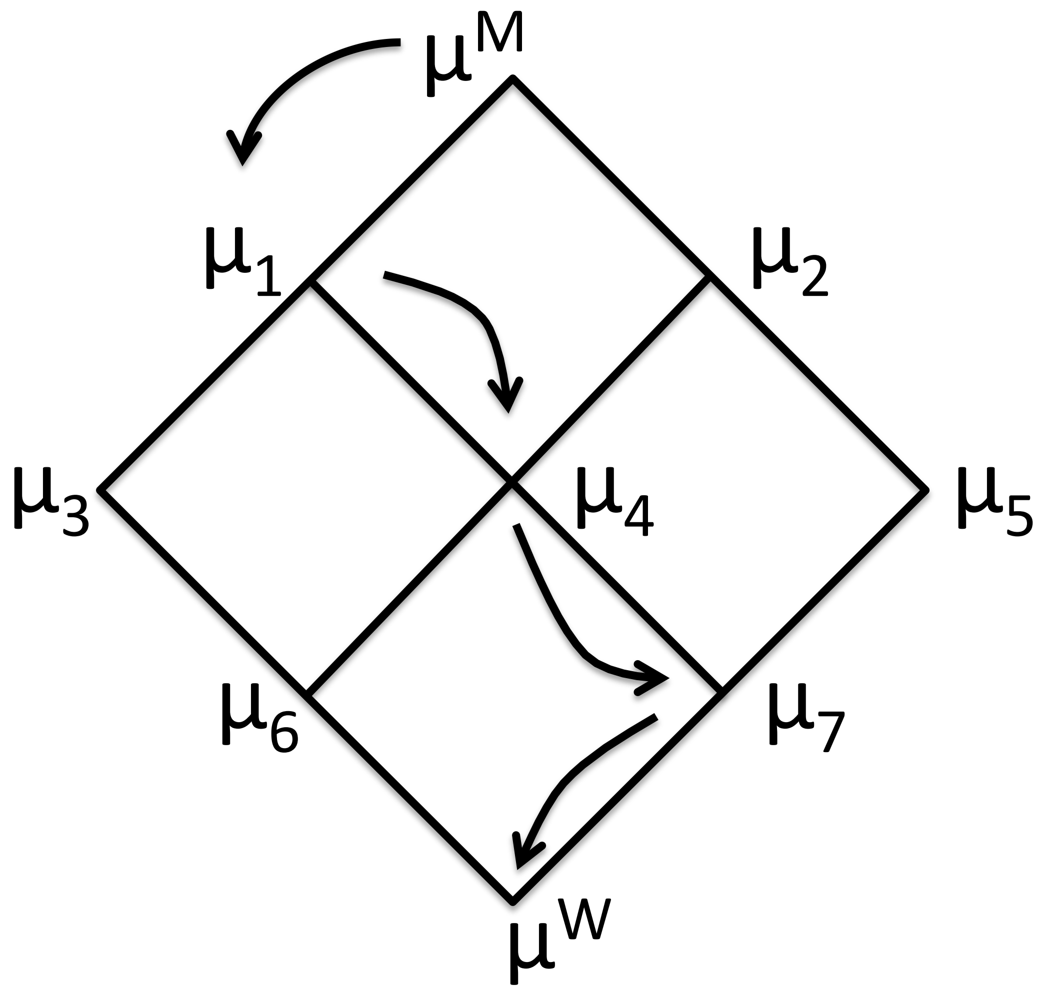
$P(m_3) = w_3, w_4, w_1, w_2$ $\quad$ $P(w_3) = m_2, m_1, m_4, m_3$

$P(m_4) = w_4, w_3, w_2, w_1$ $\quad$ $P(w_4) = m_1, m_2, m_3, m_4$

Finding All Achievable Pairs

Idea:

- Compute μ^M
- walk down lattice.



Finding All Achievable Pairs

- To walk down, rotate just one cycle
- Creates path $\mu^M = \mu^1, \ldots, \mu^k = \mu^W$ in lattice

Claim [Irving and Leather]. Any such path generates all achievable pairs.

Question. How deep is lattice?

Finding All Achievable Pairs

Claim [Irving and Leather]. Any such path generates all achievable pairs.

Prf.

- If $\mu_i(m) = w_i \neq w_{i+1} = \mu_{i+1}(m)$
- and there is achievable w with $w_i >_m w >_m w_{i+1}$,
- then can find matching μ with $\mu_i >_M \mu >_M \mu_{i+1}$.

Finding All Achievable Mates

Given woman w,

- Run men-proposing deferred acceptance
- Find worst stable mate m of w
- Truncate w's list just before m (make m unacceptable)
- Iterate until w is single