EECS 495: Market Design
Instructor: Nicole Immorlica

Collaboration Policy: you are encouraged to work in groups, but you must write up your own solutions.

1. *Roommates Problem.*

   Suppose that there are n graduate students. We will place at most two students in an office (there are n offices and students are indifferent over offices). Each student has strict preferences over the other $n-1$ students and having a one person office (being matched with herself). This problem is known as the roommates problem. Show that there may not exist a pairwise-stable matching in a roommates problem.

2. *Marriage with Dogs.*

   Suppose that there are n men, n women, and and n dogs. A matching is a set of 3-tuples (m, w, d) such that m is a man, w is a woman and d is a dog, such that every individual is included in one and only one triple. For any matching μ, if triple $(m, w, d) \in \mu$ then we say that m, w, d are matched with each other under μ. Each individual has preferences of pairs of partners: for example a dog has preferences over its men and women pairs (i.e. owners) etc. (assume that being unmatched and being matched with one partner are stirrctly worse than being matched with two partners for each individual – woman, dog or man. A matching is stable if there is no three-way deviation (i.e. no triple of a man, a woman and a dog such that each of them would strictly like to be matched with the other two rather than her current partners).

   (a) Show that a stable matching may not exist in a three-sided matching problem.

   (b) *Optional.* The following is an open question (as far as I know): Suppose preferences of the individuals are restricted as follows: Dogs have preferences only over men (they are indifferent over women), men have preferences over only women (they are indifferent over dogs) and women have preferences over only dogs (they are indifferent over men). Show that there always exists a stable matching in this problem. This question is answered positively for $n = 3, 4$.

3. *Faithfulness in Matching Markets.*

   Let (M, W, P) be a marriage market. A pair (m, w) M-faithfully blocks a matching μ if $m >_w \mu(m)$, $w >_m \mu(m)$, and $\mu(m) = m$. A matching is M-faithful if it is IR and there are no M-faithful blockings. (Note there can be non-M-faithful blockings.) Prove or disprove the following statements:

   (a) There always exists an M-faithful matching.

   (b) There exists a men-optimal M-faithful matching (i.e., an M-faithful matching weakly preferred by every man to every other M-faithful matching).

(c) The set of M-faithful matchings forms a lattice (i.e., define the join of two M-faithful matchings as the function in which every man receives the best partner and every woman receives the worst and argue whether this is an M-faithful matching).

4. *Median Stable Matchings.*

   Recall we saw in lecture that median stable matchings exist. We proved this using the linear programming representation for stable matchings. Find an alternate proof of this fact using the lattice structure of stable matchings.

5. *Equilibria in Marriage Games*

   Let (M, W, P) be a marriage market. Consider the following strategic-form game: Every man and woman knows the preferences of the others. Every man simultaneously announces the name of a woman or his own name and at the same time every woman simultaneously announces the name of a man or her own name. If a man and woman announce each other's name, they are matched with each other, otherwise they remain unmatched. Let $\mu(s)$ be the matching generated by this game when each agent i announces s_i and $s = (s_i)_{i \in M \cup W}$. Suppose that preferences allow indifferences.

   (a) A strong Nash-equilibrium is a strategy profile s such that there is no coalition $C \subseteq M \cup W$ and strategy profile s'_C for these agents with $\mu(s'_C, s_{-C})(i) >_i \mu(s)(i)$ for all $i \in C$. Prove that the set of strong Nash-equilibrium matchings is equal to the set of stable matchings.

   (b) Can you characterize the set of Nash equilibrium matchings? (What are the common properties of the matchings that can be sustained at a Nash equilibrium)?

6. *Manipulation via Capacities.*

   Find a many-to-one stable matching setting in which hospitals can manipulate and improve their allocation (for some consistent cardinal utilities) only by manipulating capacity constraints.

7. *Serial dictatorship and relation to Pareto-efficiency.*

   (a) Prove that for every house allocation market, and for every Pareto-efficient matching in this market, there exists a serial dictatorship mechanism that achieves it.

   (b) Find another mechanism (not serial dictatorship) which is both Pareto-efficient and strategyproof.

8. *Bossiness and manipulability in housing markets.*

   (a) Let ϕ be a house allocation mechanism. Prove that ϕ is coalitional-proof if and only if it is non-bossy and strategy-proof.

   (b) Find a bossy, strategy-proof and neutral house allocation mechanism. Also find a non-bossy, manipulable (non-strategy-proof) and neutral house allocation mechanism.

9. *The Core of Housing Markets.*

   Recall in lecture we saw the core of housing markets is non-empty and the top trading cycles algorithm outputs a point in the core. In this problem you will explore properties of the core.

   (a) Show that there is a *unique* point in the core of the housing market.

   (b) Find an example that demonstrates the core might not exist in housing markets with indifferences in preference lists.

   (c) Suppose agents are initially endowed with three houses each, and agents have strict preferences over sets of houses. Find an example that demonstrates the core might not exist in such markets.