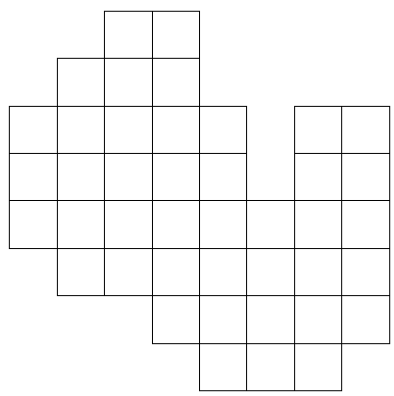
CS286r: Matching and Market Design
Instructor: Nicole Immorlica
Problem Set #1
Due: March 14, 2014

1. (10 points) Prove that if set S is covered in some matching, then it is covered in a maximum matching. We say that S is covered by a set of edges if every vertex in S is incident to at least one edge of the set.

2. (15 points) Consider the problem of perfectly tiling a grid with dominoes. Each domino covers two adjacent grid squares. A tiling is an arrangement of dominoes such that no two overlap and no domino exits the region defined by the grid. A tiling is perfect if every grid square is covered.

   (a) (5 points) Show that this problem can be formulated as the problem of deciding whether a bipartite graph has a perfect matching.

   (b) (10 points) Can the following grid be tiled by dominoes? Give a tiling or a short proof that no tiling exists.



3. (15 points) In class we showed that for combinatorial markets with unit-demand agents, a Walrasian equilibrium (WE) always exists. In this problem we will generalize this result. We say valuation function V is **gross substitutes** if the following is true: for all pairs of price vectors $\bar{p}$ and $\bar{q}$ with $q_i \geq p_i$ for all i, and for any demanded set $A \in \underset{S}{argmax} \left\{ V(S) - \sum_{j \in S} p_j \right\}$, there is a demanded set $B \in \underset{S}{argmax} \left\{ V(S) - \sum_{j \in S} q_j \right\}$ such that $\{a \in A | \, p_a = q_a\} \subseteq B$.

   (a) (5 points) Prove that all unit-demand valuations are gross substitutes.

   (b) (10 points) Prove that a combinatorial market with gross substitutes valuations always has a WE.

   Hint: Describe a tâtonnement process with an appropriate tie-breaking rule, and argue that it converges to an ϵ-approximate WE. Then take ϵ to 0.

4. (10 points) For each of the following combinatorial markets give a WE or prove that no WE exists. (Note: $V(\emptyset) = 0$ in all valuation functions.)

   (a) (1 point) Items: $\{a, b\}$, agents: $\{1, 2\}$.
   $V_1(\{a, b\}) = 7$, $V_1(\{a\}) = V_1(\{b\}) = 0$
   $V_2(\{a, b\}) = V_2(\{a\}) = V_2(\{b\}) = 3$

   (b) (2 points) Items: $\{a, b, c, d\}$, agents: $\{1, 2\}$.
   $V_1(S) = \sqrt{\text{—S—}}$ for all $S \subseteq \{a, b, c, d\}$
   $V_2(S) = \frac{1}{2}|S|$ for all $S \subseteq \{a, b, c, d\}$

   (c) (3 points) Items: $\{a, b, c\}$, agents: $\{1, 2, 3\}$.
   $$V_1(S) = \begin{cases} 3 & \{a, b\} \subseteq S \\ 1 & c \in S \, and \, \{a, b\} \nsubseteq S \\ 0 & else \end{cases}$$
   $$V_2(S) = \begin{cases} 3 & \{b, c\} \subseteq S \\ 1 & a \in S \, and \, \{b, c\} \nsubseteq S \\ 0 & else \end{cases}$$
   $$V_3(S) = \begin{cases} 3 & \{a, c\} \subseteq S \\ 1 & b \in S \, and \, \{a, c\} \nsubseteq S \\ 0 & else \end{cases}$$

   (d) (4 points) Items: $\{a, b, c, d\}$, agents: $\{1, 2, \}$.
   $V_1(S) = \frac{7}{8}$ for all $S \neq \emptyset$
   $V_2(S) = \max\{|S|, 2\}$ for all $S \neq \emptyset$

5. (15 points) Let G = (V,E) be a bipartite graph. Prove that the matching polytope $P_{matching}(G)$ is equal to the set of non-negative vectors $x \in \mathbb{R}^E$ satisfying: $\sum_{v \in e} x_e \leq 1$ for each $v \in V$.

   Hint: While it is possible to solve this problem using total unimodularity, you can also do it via reduction to the perfect matching polytope we saw in lecture.

6. (10 points) Prove or disprove each inference: Ex-ante efficiency $\iff$ Ex-post efficiency $\iff$ Ordinal efficiency. (A lottery is **ex-post efficient** iff it assigns positive weight to only Pareto efficient assignments. A random assignment P is **ordinally efficient** given preferences $\succcurlyeq$, iff it is not stochastically dominated by any other random assignment under $\succcurlyeq$. A feasible allocation is said to be **ex-ante efficient** if there does not exist a feasible (random) allocation that is preferred by all agents.

7. (25 points) Select one of the papers listed in the exercises tab of the course website and write a 1-page typed critique that summarizes the main contributions, discusses the significance of the contributions, and discusses the implications for markets.