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Incentives in the probabilistic serial mechanism ☆

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Abstract

The probabilistic serial mechanism (Bogomolnaia and Moulin, 2001 [9]) is ordinally efficient but not strategy-proof. We study incentives in the probabilistic serial mechanism for large assignment problems. We establish that for a fixed set of object types and an agent with a given expected utility function, if there are sufficiently many copies of each object type, then reporting ordinal preferences truthfully is a weakly dominant strategy for the agent (regardless of the number of other agents and their preferences). The non-manipulability and the ordinal efficiency of the probabilistic serial mechanism support its implementation instead of random serial dictatorship in large assignment problems.

JEL classification: C70; D61; D63

Keywords: Random assignment; Probabilistic serial mechanism; Ordinal efficiency; Exact strategy-proofness in large markets; Random serial dictatorship

1. Introduction

In an assignment problem a set of indivisible objects that are collectively owned must be allocated to a number of agents, who can each consume at most one object. University house allocation and student placement in public schools are examples of important assignment prob-

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lems.[1] The allocation mechanism needs to be fair and efficient. In many applications monetary transfers are precluded and fairness concerns motivate random assignments. Often the allocation must be based on the agents' reports of ordinal preferences over objects rather than cardinal preferences, as elicitation of cardinal preferences may prove difficult.[2]

There are two important solutions to the random assignment problem: the random serial dictatorship mechanism [3] and the probabilistic serial mechanism [9]. Random serial dictatorship draws each possible ordering of the agents randomly (with equal probability) and, for each realization of the ordering, assigns the first agent his most preferred object, the next agent his most preferred object among the remaining ones, and so on. This mechanism is strategy-proof and ex post efficient. Random serial dictatorship is used for house allocation in universities and for student placement in public schools.

Despite its ex post efficiency, random serial dictatorship may result in unambiguous efficiency loss ex ante. Bogomolnaia and Moulin [9] provide an example in which the random serial dictatorship assignment is first-order stochastically dominated by another random assignment with respect to the ordinal preferences of every agent. A random assignment is called ordinally efficient if it is not first-order stochastically dominated with respect to the ordinal preferences of every agent by any other random assignment. Clearly, any ordinally efficient random assignment is ex post efficient. Ordinal efficiency is a suitable efficiency concept in the context of allocation mechanisms based solely on ordinal preferences.

Bogomolnaia and Moulin propose the probabilistic serial mechanism as an alternative to random serial dictatorship. The idea is to regard each object as a continuum of "probability shares." Each agent "eats" his most preferred available object with speed one at every point in time between 0 and 1. The probabilistic serial (random) assignment is defined as the profile of shares of objects that agents eat by time 1. The ensuing random assignment is ordinally efficient and envy-free with respect to the reported preferences.

However, the desirable properties of the probabilistic serial mechanism come at a cost. The mechanism is not strategy-proof, which means that in some circumstances an agent can obtain a more preferred random assignment (with respect to his true expected utility function) by misstating his ordinal preferences. When agents report false preferences, the probabilistic serial assignment is not necessarily ordinally efficient or envy-free with respect to the true preferences. Whether the probabilistic serial mechanism is an appropriate solution to the random assignment problem has been unclear due to its incentive issues.

We show that agents have incentives to report their ordinal preferences truthfully in the probabilistic serial mechanism if the market is sufficiently large. More specifically, our main result is that, for a fixed set of object types and an agent with a given expected utility function over these objects, if the number of copies of each object type is sufficiently large, then truthful reporting of ordinal preferences is a weakly dominant strategy for the agent (for any set of other participating agents and their preferences). The incentive compatibility of the probabilistic serial mechanism we discover, together with its better efficiency and fairness properties, supports its use rather than the random serial dictatorship mechanism in large allocation problems.

[1] See Abdulkadiroğlu and Sönmez [4] and Chen and Sönmez [12] for house allocation, and Balinski and Sönmez [7] and Abdulkadiroğlu and Sönmez [6] for student placement. Practical considerations in designing student placement mechanisms in New York City and Boston are discussed by Abdulkadiroğlu, Pathak and Roth [1] and Abdulkadiroğlu, Pathak, Roth, and Sönmez [2].

[2] The market-like mechanism of Hylland and Zeckhauser [17] is one of the few solutions proposed for the random assignment problem where agents report cardinal preferences.

We develop a lower bound on the supply of each object type sufficient for truth-telling to be a weakly dominant strategy for an agent. We show by example that the bound cannot be improved by a factor greater than $x \approx 1.76322$.

In our setting the large market assumption entails the existence of a large supply of each object type. This assumption is satisfied by several interesting models. For instance, the "replica economy" model often used to discuss asymptotic properties of markets is a special case of our setting (since the number of copies of each object type is large in an economy that is replicated many times). Also, the assumption is natural in applications. In the context of university housing, rooms may be divided into several categories according to building and size; all rooms in the same category are considered identical.[3] In the case of student placement in public schools, there are typically many identical seats at each school. As an illustration, consider a school choice setting where a student finds only 10 schools acceptable, and his utility difference between any two consecutively ranked schools is constant. Our main result implies that if there are at least 18 seats at every school, then truth-telling is a weakly dominant strategy for the student in the probabilistic serial mechanism.

1.1. Related literature

Manea [26] establishes that the fraction of preference profiles for which the random serial dictatorship assignment is ordinally efficient vanishes in large allocation problems. This result provides additional support to the use of the probabilistic serial mechanism. Simulations based on real preferences also suggest that the probabilistic serial mechanism achieves an efficiency gain over random serial dictatorship in large markets. Using the data of student placement in public schools in New York City, Pathak [28] compares the resulting random allocations for each student under the two mechanisms in terms of first-order stochastic dominance. He finds that about 50% of the students are better off under the probabilistic serial mechanism, while about 6% are better off under the random serial dictatorship mechanism.[4]

Che and Kojima [11] prove that the assignments in the probabilistic serial and random serial dictatorship mechanisms converge to the same limit as the supply of each object goes to infinity. Hence the magnitude of the efficiency loss under random serial dictatorship may diminish in large allocation problems. However, the two mechanisms are equivalent only asymptotically, and the paper does not analyze the speed of convergence to the common limit. By contrast, our paper shows incentive compatibility of the probabilistic serial mechanism in a large but finite allocation problem, and offers a lower bound on the size of the problem which is sufficient for this conclusion.

Incentive properties in large markets have been investigated in various areas of economics. For pure exchange economies, Roberts and Postlewaite [29] show that agents have diminishing incentives to misrepresent demand functions in the competitive mechanism as the market becomes large. Similarly, in the context of double auctions, Gresik and Satterthwaite [16], Rustichini, Satterthwaite, and Williams [31], and Cripps and Swinkels [14] show that equilibrium behavior converges to truth-telling as the number of traders grows. In the two-sided matching setting, Roth and Peranson [30], Immorlica and Mahdian [18] and Kojima and Pathak [24] show that the

[3] For example, the assignment of rooms in Harvard graduate dorms is based only on preferences over eight types of rooms—there are two possible room sizes in each of four buildings.

[4] For the rest of the students, the random allocations corresponding to the two mechanisms are not comparable in terms of first-order stochastic dominance.

deferred acceptance algorithm proposed by Gale and Shapley [15] is difficult to manipulate profitably when the number of participants become large. Most of this research shows either that the gain from manipulation converges to zero or that equilibrium behavior converges to truth-telling in the limit as the market becomes large. In contrast to these "approximate" and "asymptotic" results, we show that truth-telling is an exact weakly dominant strategy in the probabilistic serial mechanism for finitely large markets.[5]

There is a growing literature on random assignment and ordinal efficiency. Abdulkadiroğlu and Sönmez [5] provide a characterization of ordinal efficiency based on the idea of dominated sets of assignments. McLennan [27] proves that any random assignment which is ordinally efficient for some ordinal preferences is welfare-maximizing with respect to some expected utility functions consistent with the ordinal preferences. A short constructive proof is offered by Manea [25]. Kesten [22] introduces the top trading cycles from equal division mechanism, and shows that it is equivalent to the probabilistic serial mechanism. The probabilistic serial mechanism is extended to cases with non-strict preferences, existing property rights, and multi-unit demands by Katta and Sethuraman [21], Yilmaz [33], and Kojima [23], respectively. On the restricted domain of the scheduling problem, Crès and Moulin [13] show that the probabilistic serial mechanism is group strategy-proof and first-order stochastically dominates the random serial dictatorship mechanism, and Bogomolnaia and Moulin [10] find two characterizations of the probabilistic serial mechanism.

The rest of the paper is organized as follows. Section 2 describes the model. The main result is presented in Section 3, with the proof relegated to Appendix B. Section 4 provides a detailed example and Section 5 concludes.

2. Model

A *random assignment problem* is a quadruple $\Gamma = (N, (\succ_i)_{i\in N}, \hat{O}, (q_a)_{a\in\hat{O}})$. N and $\hat{O}$ represent (finite) sets of *agents* and *proper object types*, respectively. The *quota* (number of copies) of object a is denoted by q_a. There is an infinite supply of a *null object* $\emptyset$ (which does not belong to $\hat{O}$), $q_\emptyset = \infty$. Each agent $i \in N$ has a *strict preference* $\succ_i$ over $O := \hat{O} \cup \{\emptyset\}$. We write $a \succeq_i b$ if and only if $a \succ_i b$ or $a = b$. When N is fixed, $\succ$ denotes $(\succ_i)_{i\in N}$ and $\succ_{N'}$ denotes $(\succ_i)_{i\in N'}$ (for $N' \subset N$).

A *deterministic assignment* for the problem Γ is a matrix $X = (X_{ia})$, with rows indexed by $i \in N$ and columns by $a \in O$, such that $X_{ia} \in \{0, 1\}$ for all i and a, $\sum_{a\in O} X_{ia} = 1$ for all i, and $\sum_{i\in N} X_{ia} \leqslant q_a$ for all a. The value of X_{ia} is 1 (0) if agent i receives (does not receive) object a at the assignment X. Hence the constraints $\sum_{a\in O} X_{ia} = 1$ and $\sum_{i\in N} X_{ia} \leqslant q_a$ mean that i receives exactly one object and at most q_a agents receive a at the assignment X.

A *lottery assignment* is a probability distribution w over the set of deterministic assignments, where $w(X)$ denotes the probability of the assignment X. A *random assignment* is a matrix $P = (P_{ia})$, with $P_{ia} \geqslant 0$ for all i and a, $\sum_{a\in O} P_{ia} = 1$ for all i, and $\sum_{i\in N} P_{ia} \leqslant q_a$ for all a; P_{ia} stands for the probability that agent i receives object a. A lottery assignment w *induces* the random assignment $\sum_X w(X)X$. The entry (i, a) in this matrix represents the probability

[5] However, Jackson [19] notes that truth-telling becomes a weakly dominant strategy in the competitive mechanism for large economies when agents are constrained to report from a finite set of demand functions. In contrast, Jackson and Manelli [20] show that Nash equilibrium behavior in the competitive mechanism need not converge to truth-telling in large economies with unrestricted demand functions.

that agent i is assigned object a under w. The following proposition is a generalization of the Birkhoff–von Neumann theorem [8,32].

Proposition 1. *Every random assignment can be written as a convex combination of deterministic assignments.*[6]

The proof is in Appendix A. By Proposition 1, any random assignment is induced by a lottery assignment. Henceforth, we identify lottery assignments with the corresponding random assignments and use these terms interchangeably.

We assume that each agent has a *von Neumann–Morgenstern expected utility function* over random assignments. The *utility index* of agent i is a function $u_i : O \to \mathbb{R}$. We extend the domain of u_i to the set of random assignments as follows. Agent i's expected utility for the random assignment P is $u_i(P) = \sum_{a\in O} P_{ia} u_i(a)$. We say that u_i is *consistent* with $\succ_i$ when $u_i(a) > u_i(b)$ if and only if $a \succ_i b$.

A random assignment P *ordinally dominates* another random assignment P' *at* $\succ$ if

$$\sum_{b \succeq_i a} P_{ib} \geqslant \sum_{b \succeq_i a} P'_{ib}, \quad \forall i \in N,\ \forall a \in O,$$

with strict inequality for some i, a. A random assignment is *ordinally efficient at* $\succ$ if it is not ordinally dominated at $\succ$ by any other random assignment. Suppose that P ordinally dominates P' at $\succ$. Then each agent i weakly prefers P to P' in terms of first-order stochastic dominance with respect to $\succ_i$. Equivalently, every agent i weakly prefers P to P' according to any expected utility function consistent with $\succ_i$.

We extend the *probabilistic serial* mechanism proposed by Bogomolnaia and Moulin [9] to our setting. Each object is viewed as a divisible good of "probability shares." Each agent "eats" his most preferred available object with speed one at every time $t \in [0, 1]$—object a is *available* at time t if less than q_a share of a has been eaten away by t. The resulting profile of object shares that agents eat by time 1 corresponds to a random assignment, which is the *probabilistic serial* (*random*) *assignment*.

Formally, the (*symmetric simultaneous*) *eating algorithm* defines the probabilistic serial assignment for the preference profile $\succ$ as follows. For any $a \in O' \subset O$, let $N(a, O') = \{i \in N \mid a \succeq_i b,\ \forall b \in O'\}$ represent the set of agents whose most preferred object in O' is a. Set $O^0 = O$, $t^0 = 0$, and $P^0_{ia} = 0$ for every $i \in N$ and $a \in O$. For all $v \geqslant 1$, given $O^0, t^0, (P^0_{ia}), \dots, O^{v-1}, t^{v-1}, (P^{v-1}_{ia})$, define

$$t^v = \min_{a \in O^{v-1}} \max\left\{ t \in [0,1] \;\middle|\; \sum_{i \in N} P^{v-1}_{ia} + \left|N\left(a, O^{v-1}\right)\right|\left(t - t^{v-1}\right) \leqslant q_a \right\},$$

$$O^v = O^{v-1} \setminus \left\{ a \in O^{v-1} \;\middle|\; \sum_{i \in N} P^{v-1}_{ia} + \left|N\left(a, O^{v-1}\right)\right|\left(t^v - t^{v-1}\right) = q_a \right\},$$

$$P^v_{ia} = \begin{cases} P^{v-1}_{ia} + t^v - t^{v-1} & \text{if } i \in N(a, O^{v-1}), \\ P^{v-1}_{ia} & \text{otherwise,} \end{cases}$$

[6] There may be multiple convex combinations of deterministic assignments that induce the same random assignment.

where for any set S, $|S|$ denotes the cardinality of S. Since $\hat{O}$ is a finite set, there exists $\bar{v}$ such that $t^{\bar{v}} = 1$. We define $PS(\succ) := P^{\bar{v}}$ as the probabilistic serial assignment for the preference profile $\succ$.

The intuition for the recursive definition above is as follows. Stage $v = 1, \dots$ of the eating algorithm begins at time t^{v-1} with share $\sum_{i \in N} P_{ia}^{v-1}$ of object $a \in O$ having been eaten already. O^{v-1} denotes the set of object types that have not been completely consumed by time t^{v-1}. Each agent in $N(a, O^{v-1})$ eats a, which is his most preferred object in O^{v-1}, until its entire quota q_a is consumed.

Bogomolnaia and Moulin [9] establish that the probabilistic serial assignment is ordinally efficient and envy-free in their setting (a random assignment is *envy-free* if every agent weakly prefers his own assignment to that of any other agent in terms of first-order stochastic dominance with respect to his reported ordinal preferences). The proofs can be easily adapted to our setting. Neither ordinal efficiency nor envy-freeness is satisfied by the extensively used *random serial dictatorship* mechanism [3], also known as the *random priority* mechanism [9].

However, the high degree of efficiency and fairness of the probabilistic serial mechanism is not without cost. The mechanism is *not strategy-proof*, that is, an agent is sometimes better off misstating his preferences. In fact, a result of Bogomolnaia and Moulin [9] implies that there is no mechanism satisfying strategy-proofness, ordinal efficiency and envy-freeness. The ordinal efficiency and envy-freeness of the probabilistic serial mechanism are based on the presumption that agents report their ordinal preferences truthfully. If agents misreport preferences, then neither of the two desirable properties is guaranteed. Therefore, it is important to identify conditions under which agents have incentives to report their ordinal preferences truthfully in the probabilistic serial mechanism.

3. Result

We show that agents have incentives to report ordinal preferences truthfully in the probabilistic serial mechanism when the quota of each object is sufficiently large.

Theorem 1. *Let u_i be an expected utility function consistent with a preference $\succ_i$.*

(i) *There exists M such that if $q_a \geqslant M$ for all $a \in \hat{O}$, then*

$$u_i\big(PS(\succ_i, \succ_{N\setminus\{i\}})\big) \geqslant u_i\big(PS\big(\succ_i', \succ_{N\setminus\{i\}}\big)\big)$$

for any preference $\succ_i'$, any set of agents $N \ni i$, and any preference profile $\succ_{N\setminus\{i\}}$.

(ii) *Claim* (i) *is satisfied for $M = xD/d$, where $x \approx 1.76322$ solves $x\ln(x) = 1$, $D = \max_{a \succeq_i b \succeq_i \emptyset} u_i(a) - u_i(b)$, and $d = \min_{a \succ_i b,\, a \succeq_i \emptyset} u_i(a) - u_i(b)$.*

A formal proof of the theorem is presented in Appendix B. For a sketch of the argument, fix a preference profile $\succ$, and denote by $\succ' = (\succ_i', \succ_{N\setminus\{i\}})$ the preference profile where agent i reports $\succ_i'$ instead of $\succ_i$. By deviating from $\succ_i$ to $\succ_i'$, agent i may influence the outcome of the eating algorithm through the following two channels:

- at any instance in the algorithm, *for a fixed set of available objects*, reporting $\succ_i'$ may prevent i from eating his $\succ_i$-most preferred available object;

- reporting $\succ_i'$ can *influence the availability schedule of the objects*, e.g., reporting an object as less desirable may lengthen the period when it is available, and further affect the eating behavior of other agents, which in turn can change the times when other objects are available.

The former channel is always detrimental to i, but the latter may be favorable. We prove that i's benefit from the latter channel is smaller than his cost from the former when the quota of each object becomes large.

More specifically, suppose that over some time interval $[t, t')$ agent i eats object a under $\succ'$ and object b under $\succ$, and $a \succ_i b$. It must be that a is not available under $\succ$ at t (otherwise i would not be eating b). The proof shows that the share of a available at t under $\succ'$ is small. Since a large part of the q_a share of a is consumed under $\succ'$ before t, if q_a is large, then many agents must eat a over $[t, t')$ under $\succ'$. Hence a cannot be available under $\succ'$ long after t. Therefore, the interval $[t, t')$ must be short. We establish that the size of the interval $[t, t')$ is of an order of magnitude smaller than the sum, denoted by λ, of the lengths of time intervals on which agent i's consumption in the eating algorithm under $\succ$ is $\succ_i$-preferred to that under $\succ'$.

Suppose that $q_a \geqslant M$ for all $a \in \hat{O}$. In Sections B.2 and B.3 of Appendix B we find lower bounds on M sufficient for truth-telling to be a weakly dominant strategy for agent i. Let $k = |\{a \in \hat{O} \mid a \succ_i \emptyset\}|$ denote the number of object types that are $\succ_i$-preferred to the null object.

Section B.2 provides a rough bound. Based on the intuition above, we show that the sum of the lengths of time intervals on which i benefits from reporting $\succ_i'$ rather than $\succ_i$ does not exceed $\lambda((1 + 1/M)^k - 1)$. Hence i's expected utility gain from misreporting preferences over these intervals is at most $D\lambda((1 + 1/M)^k - 1)$. At the same time, i's expected utility loss over the intervals where his consumption under $\succ$ is $\succ_i$-preferred to that under $\succ'$ is at least $d\lambda$. Therefore,

$$u_i\big(PS(\succ)\big) - u_i\big(PS(\succ')\big) \geqslant d\lambda - D\lambda\left(\left(1 + \frac{1}{M}\right)^k - 1\right).$$

The right-hand side of the latter inequality is non-negative, and hence truth-telling is a weakly dominant strategy for agent i, if

$$M \geqslant (k + 1)\frac{D}{d}. \tag{1}$$

Section B.3 refines the bound. Let

$$\Lambda = \frac{\lambda}{M}\left(1 + \frac{1}{M}\right)^{k-1}.$$

The key observation is that the object i eats at any time t under $\succ$ is $\succeq_i$-preferred to that he eats at $t + \Lambda$ under $\succ'$. Then we can evaluate i's expected utility gain from reporting $\succ_i'$ rather than $\succ_i$ using a translation by Λ of his eating schedule under $\succ$ with respect to that under $\succ'$. We show that i's benefit from misreporting preferences does not exceed the integral of the utility difference between his $\succ_i$-most preferred object and his consumption under $\succ$ over the time interval $[1 - \Lambda, 1]$. This leads to a bound on i's expected utility gain from misreporting preferences of $D\Lambda$. Hence,

$$u_i\big(PS(\succ)\big) - u_i\big(PS(\succ')\big) \geqslant d\lambda - D\frac{\lambda}{M}\left(1 + \frac{1}{M}\right)^{k-1}.$$

A sufficient condition for the right-hand side of the inequality above be non-negative, and truth-telling be a weakly dominant strategy for agent i, is

$$M \geqslant x\frac{D}{d}. \tag{2}$$

Note that the upper bound on i's expected gain from misreporting his preferences is of order $D\lambda/M$ in Section B.3, but of order $D\lambda k/M$ in Section B.2. Consequently, (2) provides a weaker sufficient condition than (1).

Clearly, $D/d \geqslant k$, and if the utility difference between no two consecutively ranked objects varies substantially, then D/d is close to k. For instance, consider a school choice setting where student i finds only 10 schools acceptable, and his utility difference between any two consecutively ranked schools is constant. In this case $D/d = 10$. If there are at least 18 seats at every school, then truth-telling is a weakly dominant strategy for i in the probabilistic serial mechanism.

One important feature of the bound (2) is that it is independent of the misstated ordinal preferences $\succ_i'$, the set of agents $N \setminus \{i\}$ and their preference profile $\succ_{N\setminus\{i\}}$. In particular, agent i can verify whether (2) holds using only his information about D/d. Therefore, whenever (2) holds, truth-telling is a best response for i in the probabilistic serial mechanism independently of how many other agents participate and what preferences they report. Even when the quotas are not sufficiently large to make truth-telling a weakly dominant strategy for all agents, truth-telling may be a weakly dominant strategy for some of them.

In the statement of Theorem 1 the condition "$q_a \geqslant M$ for all $a \in \hat{O}$" can be replaced with "$q_a \geqslant M$ for all $a \succ_i \emptyset$." Theorem 1 has the following corollary.

Corollary 1. *Suppose that the set $\hat{O}$ of proper object types and the set $\mathcal{U}$ of expected utility functions on lotteries over $\hat{O} \cup \{\emptyset\}$ are fixed and finite. There exists M such that if $q_a \geqslant M$ for all $a \in \hat{O}$, then for any set of participating agents, truth-telling is a weakly dominant strategy in the probabilistic serial mechanism for every agent whose utility function is in $\mathcal{U}$.*

Corollary 1 implies that the probabilistic serial mechanism becomes strategy-proof in large allocation problems where the expected utility functions of all agents belong to a given finite set. The latter assertion includes the special case of replica economies. Consider a problem $\Gamma = (N, (\succ_i)_{i\in N}, \hat{O}, (q_a)_{a\in\hat{O}})$ and an expected utility u_i consistent with $\succ_i$ for each i in N. For any positive integer M, the *M-fold replica economy* of $(\Gamma, (u_i)_{i\in N})$ is a random assignment problem in which there are M "replicas" of each agent i with a common utility function u_i, and there are Mq_a copies of each object a in $\hat{O}$. A consequence of the assertion above is that for sufficiently large M, truth-telling is a weakly dominant strategy for every agent in the probabilistic serial mechanism for the M-fold replica of $(\Gamma, (u_i)_{i\in N})$.

4. Example

We present an example that serves three purposes. First, it illustrates some of the ideas of the proof of Theorem 1. Second, it shows that the bound from part (ii) of Theorem 1 cannot be improved by a factor greater than $x \approx 1.76322$. Third, it shows that the conclusion of the theorem cannot be strengthened to claim the existence of M such that if $q_a \geqslant M$ for all $a \in \hat{O}$, then truth-telling is a weakly dominant strategy for agent i in the probabilistic serial mechanism for every expected utility function u_i. That is, the order of quantifiers $\forall u_i, \exists M$ cannot be replaced with $\exists M, \forall u_i$.

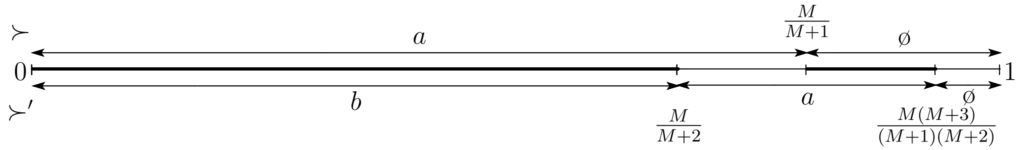

Fig. 1. Eating schedules for agent i under $\succ$ and $\succ'$.

Consider a setting with 2 types of proper objects, a and b, each having quota M. Fix $D > d > 0$ and an agent i with utility index u_i given by $u_i(a) = D, u_i(b) = D - d, u_i(\emptyset) = 0$. Note that u_i is consistent with the ordinal preference $\succ_i$ specified by $a \succ_i b \succ_i \emptyset$. Denote by $\succ_i'$ the preference for agent i with $b \succ_i' a \succ_i' \emptyset$.

Let $N = \{i\} \cup N' \cup N''$ be the set of agents, with N' and N'' of cardinalities M and $M + 1$, respectively. Assume that the preferences of the agents in $N' \cup N''$ are as follows:

$$a \succ_j \emptyset \succ_j b, \quad \forall j \in N',$$
$$b \succ_j \emptyset \succ_j a, \quad \forall j \in N''.$$

Suppose that each agent $j \neq i$ reports $\succ_j$ in the eating algorithm. If i reports $\succ_i$, then he eats object a in the time interval $[0, M/(M+1))$ and the null object in $[M/(M+1), 1]$. If i reports $\succ_i'$ instead of $\succ_i$, then he eats b in $[0, M/(M+2))$, a in $[M/(M+2), M(M+3)/(M+1)(M+2))$, and then $\emptyset$ in $[M(M+3)/(M+1)(M+2), 1]$. Fig. 1 depicts the eating schedules for agent i under the preference profiles $\succ = (\succ_i)_{i \in N}$ and $\succ' = (\succ_i', \succ_{N' \cup N''})$.

In the time interval $[0, M/(M+2))$, agent i eats a under $\succ$ and b under $\succ'$. His expected utility loss from reporting $\succ_i'$ rather than $\succ_i$ over that interval is $dM/(M+2)$. In $[M/(M+1), M(M+3)/(M+1)(M+2))$, i eats $\emptyset$ under $\succ$ and a under $\succ'$. His expected utility gain from reporting $\succ_i'$ rather than $\succ_i$ over that interval is $DM/(M+1)(M+2)$. At any time outside the two intervals, i eats an identical object under $\succ$ and $\succ'$.

As Fig. 1 illustrates, by reporting $\succ_i'$ instead of $\succ_i$, agent i suffers losses over the first thick interval and reaps benefits over the second. Note that the length of the second interval is of order M times smaller than that of the first. Hence the difference in i's expected utility between reporting $\succ_i$ and $\succ_i'$ is

$$\frac{dM}{(M+1)(M+2)}\left(M + 1 - \frac{D}{d}\right).$$

It follows that truth-telling is a weakly dominant strategy for i if $M \geqslant D/d - 1$ (the assignment under any preference report other than $\succ_i$ and $\succ_i'$ is first-order stochastically dominated with respect to $\succ_i$ by that under either $\succ_i$ or $\succ_i'$). By contrast, i has incentives to report $\succ_i'$ if $M < D/d - 1$. In particular, the bound from part (ii) of Theorem 1 cannot be improved by a factor greater than x. Furthermore, there exists no M such that if $q_a \geqslant M$ for all $a \in \hat{O}$, then agent i has incentives to report $\succ_i$ for all $D > d > 0$.

There is a delicate part of the proof which is not captured in this example. The initial change in an agent's eating behavior may induce a chain effect on the availability schedule of several objects. Hence, when an agent misstates his preferences, the first interval where he suffers losses can give rise to multiple intervals where he reaps benefits. This issue is addressed by Lemmata 5, 6 and 7 in Appendix B.

5. Conclusion

Truth-telling is a weakly dominant strategy in the probabilistic serial mechanism when there is a large supply of each object type. This result offers support to the use of the mechanism in applications such as university housing and student placement in schools. A remarkable feature of our result is that truth-telling is an exact weakly dominant strategy as opposed to an "almost dominant strategy," which is common in the literature on asymptotic incentive compatibility. Moreover, for a fixed set of object types and an agent with a given expected utility function, our conclusion holds regardless of the number of other participating agents and their ordinal preferences.

The lower bound on the supply of each object type from Theorem 1 cannot be improved by a factor greater than $x \approx 1.76322$. Whether the bound can be improved to any extent is an open question. Nevertheless, our bound may be sufficiently low to make truth-telling a weakly dominant strategy in the probabilistic serial mechanism for practical allocation problems.

Appendix A. Proof of Proposition 1

Proof. Fix the set of proper object types $\hat{O}$ with corresponding quotas $(q_a)_{a\in\hat{O}}$. Consider a random assignment P for the set of agents N. Let P' be a matrix with rows corresponding to the agents in $N \cup N'$, where N' is a set of n' fictitious agents (not in N), such that $P'_{ia} = P_{ia}$ for all $i \in N$ and $a \in O$, $P'_{ja} = (q_a - \sum_{i\in N} P_{ia})/n'$ for all $j \in N'$ and $a \in \hat{O}$, and $P'_{j\emptyset} = 1 - \sum_{a\in\hat{O}} P'_{ja}$ for all $j \in N'$.[7] For sufficiently large n', all entries of the matrix P' are non-negative. Each row of P' sums to 1, and column a of P' sums to q_a for all $a \in \hat{O}$. Since all rows and columns have integer sums and each entry is non-negative, the procedure described by Hylland and Zeckhauser [17] in the section "Conduct of the Lottery" may be adapted to the current setting to find a convex decomposition of P' into deterministic assignments for the agents in $N \cup N'$. Obviously, the restriction of this convex decomposition to the agents in N induces a convex decomposition of P into deterministic assignments for the agents in N. □

Appendix B. Proof of Theorem 1

B.1. Notation

An *eating function* e describes an eating schedule for each agent, $e_i : [0, 1] \to O$ for all $i \in N$; $e_i(t)$ represents the object that agent i is eating at time t. We require that e_i be right-continuous with respect to the discrete topology on O (the topology in which all subsets are open), that is,

$$\forall t \in [0, 1), \exists \varepsilon > 0 \quad \text{such that} \quad e_i(t') = e_i(t), \quad \forall t' \in [t, t+\varepsilon).$$

For an eating function e, let $n_a(t, e)$ be the number of agents eating from object a at time t and $\nu_a(t, e)$ be the share of object a eaten away by time t, i.e.,[8]

$$n_a(t, e) = \left|\left\{i \in N \mid e_i(t) = a\right\}\right|,$$

[7] McLennan [27] uses a similar construction.

[8] It can be shown that $n_a(\cdot, e)$ is Riemann integrable.

$$\nu_a(t,e)=\int_0^t n_a(s,e)\,ds.$$

Note that $\nu_a(\cdot,e)$ is continuous.

For every preference profile $\succ$, let $e^{\succ}$ denote the eating function generated by the eating algorithm when agents report $\succ$. Formally, $e_i^{\succ}(t)=a$ for $t\in[t^{v-1},t^v)$ if $i\in N(a,O^{v-1})$, for (O^v) and (t^v) constructed in the definition of the probabilistic serial mechanism.

Fix a preference profile $\succ$, and denote by $\succ'=(\succ'_i,\succ_{N\setminus\{i\}})$ the preference profile where agent i reports $\succ'_i$ instead of $\succ_i$. Let $\bar{e}$ be the eating function such that

$$\bar{e}_i(t)=\begin{cases} e_i^{\succ}(t) & \text{if } e_i^{\succ}(t)=e_i^{\succ'}(t),\\ \emptyset & \text{otherwise,}\end{cases}$$

and at each instance, under $\bar{e}_j$ agent $j\neq i$ is eating from his most preferred object at speed 1 among the ones still available (accounting for agent i's specified eating function $\bar{e}_i$). Note that $\bar{e}_j$ may diverge from $e_j^{\succ}$ or $e_j^{\succ'}$ for $j\neq i$ since the available objects at each time may vary across $\bar{e}$, $e^{\succ}$ and $e^{\succ'}$ due to the different eating behavior adopted by i.

Let $\beta(t),\gamma(t)$, and $\delta(t)$ denote the sums of the lengths of time intervals, before time t, on which agent i's consumption in the eating algorithm is $\succ_i$-preferred, $\succ_i$-less preferred, and different, respectively, when the reported preferences change from $\succ$ to $\succ'$. Formally,

$$\beta(t)=\int_0^t \mathbf{1}_{e_i^{\succ'}(s)\succ_i e_i^{\succ}(s)}\,ds,$$

$$\gamma(t)=\int_0^t \mathbf{1}_{e_i^{\succ}(s)\succ_i e_i^{\succ'}(s)}\,ds,$$

$$\delta(t)=\beta(t)+\gamma(t),$$

where for any logical proposition p, $\mathbf{1}_p=1$ if p is true and $\mathbf{1}_p=0$ if p is false. Set $\lambda=\gamma(1)$.

Define

$$\{a_1,a_2,\dots,a_{\bar{l}}\}=\left\{a\in\hat{O}\ \middle|\ \exists t\in[0,1),\ a=e_i^{\succ'}(t)\succ_i e_i^{\succ}(t)\right\}$$

as the set of objects that are consumed at some time under $e_i^{\succ'}$ and are $\succ_i$-preferred to the consumption at that time under $e_i^{\succ}$. The set is labeled such that $a_1\succ'_i a_2\succ'_i\cdots\succ'_i a_{\bar{l}}$. For $l=1,2,\dots,\bar{l}$, let

$$T_l=\inf\left\{t\ \middle|\ a_l=e_i^{\succ'}(t)\succ_i e_i^{\succ}(t)\right\}$$

be the first instance t when a_l is consumed under $e_i^{\succ'}$ and is $\succ_i$-preferred to the consumption at t under $e_i^{\succ}$. Clearly, $0<T_1<T_2<\cdots<T_{\bar{l}}<1$.

Let k denote the number of proper object types that are $\succ_i$-preferred to the null object, $k=|\{a\in\hat{O}\mid a\succ_i\emptyset\}|$. Note that $\bar{l}\leqslant k$ since $a_l=e_i^{\succ'}(T_l)\succ_i e_i^{\succ}(T_l)\succeq_i\emptyset$ for all l. Set $T_0=0$, $T_{\bar{l}+1}=1$ as a technical notation convention.

B.2. Part (i)

The proof uses Lemmata 1–6 below.

Lemma 1. *For all* $t \in [0,1]$ *and* $a \in \hat{O}$,

$$\nu_a\big(t, e^{\succ}\big) \geqslant \nu_a(t, \bar{e}),$$
$$\nu_a\big(t, e^{\succ'}\big) \geqslant \nu_a(t, \bar{e}).$$

Proof. By symmetry, we only need to prove the first inequality. We proceed by contradiction. Assume that there exist t and a such that $\nu_a(t, e^{\succ}) < \nu_a(t, \bar{e})$. Let

$$t_0 = \inf\big\{t \in [0,1] \,\big|\, \exists a \in \hat{O},\, \nu_a\big(t, e^{\succ}\big) < \nu_a(t, \bar{e})\big\}. \tag{3}$$

By continuity of $\nu_a(\cdot, e^{\succ}) - \nu_a(\cdot, \bar{e})$, it follows that $t_0 < 1$, and

$$\nu_a\big(t_0, e^{\succ}\big) - \nu_a(t_0, \bar{e}) \geqslant 0, \quad \forall a \in \hat{O}. \tag{4}$$

This holds trivially if $t_0 = 0$.

One consequence of (4) is that all objects that are not eaten away by time t_0 under $e^{\succ}$ cannot be eaten away by t_0 under $\bar{e}$ either. Hence the set of objects available at t_0 under $e^{\succ}$ is included in that under $\bar{e}$. It must be that if agent $j \in N$ is eating object $a \in \hat{O}$ at t_0 under $\bar{e}$ and a is available at t_0 under $e^{\succ}$, then j is eating a at t_0 under $e^{\succ}$. Formally,

$$\forall j \in N, \quad \bar{e}_j(t_0) = a \neq \emptyset \quad \text{and} \quad \nu_a\big(t_0, e^{\succ}\big) < q_a \quad \Rightarrow \quad e_j^{\succ}(t_0) = a.$$

For $j = i$ the latter step follows from the definition of $\bar{e}$. Therefore,

$$\forall a \in \hat{O}, \quad \nu_a\big(t_0, e^{\succ}\big) < q_a \quad \Rightarrow \quad n_a\big(t_0, e^{\succ}\big) \geqslant n_a(t_0, \bar{e}). \tag{5}$$

Given the right-continuity of $e^{\succ}$ and $\bar{e}$, for sufficiently small $\varepsilon > 0$, we have that for all $t \in [t_0, t_0 + \varepsilon)$ and $a \in \hat{O}$,

$$\nu_a\big(t, e^{\succ}\big) = \nu_a\big(t_0, e^{\succ}\big) + n_a\big(t_0, e^{\succ}\big)(t - t_0),$$
$$\nu_a(t, \bar{e}) = \nu_a(t_0, \bar{e}) + n_a(t_0, \bar{e})(t - t_0).$$

Using (4) and (5) we obtain $\nu_a(t, e^{\succ}) \geqslant \nu_a(t, \bar{e})$ for all $t \in [t_0, t_0 + \varepsilon)$ and $a \in \hat{O}$ with $\nu_a(t_0, e^{\succ}) < q_a$. Note that if $\nu_a(t_0, e^{\succ}) = q_a$ then the inequality $\nu_a(t, e^{\succ}) \geqslant \nu_a(t, \bar{e})$ holds trivially for all $t \geqslant t_0$.

By (3), $\nu_a(t, e^{\succ}) \geqslant \nu_a(t, \bar{e})$ for all $t \in [0, t_0)$ and $a \in \hat{O}$. The arguments above establish that $\nu_a(t, e^{\succ}) \geqslant \nu_a(t, \bar{e})$ for all $t \in [0, t_0 + \varepsilon)$ and $a \in \hat{O}$, which contradicts the definition of t_0. $\square$

Lemma 2. *For all* $t \in [0,1]$,

$$\nu_\emptyset\big(t, e^{\succ}\big) - \nu_\emptyset(t, \bar{e}) \geqslant -\delta(t).$$

Proof. Note that

$$\nu_\emptyset\big(t, e^{\succ}\big) - \nu_\emptyset(t, \bar{e}) + \delta(t) = \int_0^t \big[n_\emptyset\big(s, e^{\succ}\big) - n_\emptyset(s, \bar{e}) + \mathbf{1}_{e_i^{\succ}(s) \neq e_i^{\succ'}(s)}\big]\, ds.$$

Since $\nu_a(t, e^{\succ}) \geqslant \nu_a(t, \bar{e})$ for all $a \in \hat{O}$ and $t \in [0,1]$ by Lemma 1, an argument similar to Lemma 1 leads to

$$e_i^{\succ}(s) \neq \bar{e}_i(s) \quad \Rightarrow \quad n_\emptyset\big(s, e^{\succ}\big) \geqslant n_\emptyset(s, \bar{e}) - 1,$$
$$e_i^{\succ}(s) = \bar{e}_i(s) \quad \Rightarrow \quad n_\emptyset\big(s, e^{\succ}\big) \geqslant n_\emptyset(s, \bar{e}).$$

Thus the integrand $n_\emptyset(s, e^{\succ}) - n_\emptyset(s, \bar{e}) + \mathbf{1}_{e_i^{\succ}(s) \neq e_i^{\succ'}(s)}$ is non-negative for all $s \in [0, t]$, which completes the proof. □

Lemma 3. *For all* $t \in [0, 1]$ *and* $a \in \hat{O}$,

$$\nu_a\big(t, e^{\succ}\big) - \nu_a(t, \bar{e}) \leqslant \delta(t).$$

Proof. The inequality follows immediately from Lemmata 1 and 2, noting that

$$\sum_{a \in O} \nu_a\big(t, e^{\succ}\big) - \nu_a(t, \bar{e}) = 0, \quad \forall t \in [0, 1]. \qquad \square$$

Lemma 4. *For all* $t \in [0, 1]$ *and* $a \in \hat{O}$,

$$\nu_a\big(t, e^{\succ}\big) - \nu_a\big(t, e^{\succ'}\big) \leqslant \delta(t).$$

Proof. The inequality follows from Lemmata 1 and 3, writing

$$\nu_a\big(t, e^{\succ}\big) - \nu_a\big(t, e^{\succ'}\big) = \big[\nu_a\big(t, e^{\succ}\big) - \nu_a(t, \bar{e})\big] - \big[\nu_a\big(t, e^{\succ'}\big) - \nu_a(t, \bar{e})\big]. \qquad \square$$

Lemma 5. *For all* $l = 1, \ldots, \bar{l}$,

$$\beta(T_{l+1}) - \beta(T_l) \leqslant \frac{\delta(T_l)}{q_{a_l}}.$$

Proof. Since $a_l = e_i^{\succ'}(T_l) \succ_i e_i^{\succ}(T_l)$, it follows that the object a_l is not available at time T_l under the eating function $e^{\succ}$, i.e., $\nu_{a_l}(T_l, e^{\succ}) = q_{a_l}$. By Lemma 4,

$$\nu_{a_l}\big(T_l, e^{\succ'}\big) \geqslant \nu_{a_l}\big(T_l, e^{\succ}\big) - \delta(T_l) > q_{a_l} - 1. \tag{6}$$

As $n_{a_l}(\cdot, e^{\succ'})$ is increasing on the time interval where a_l is available under $e^{\succ'}$,

$$n_{a_l}\big(T_l, e^{\succ'}\big) > n_{a_l}\big(T_l, e^{\succ'}\big)T_l \geqslant \int_0^{T_l} n_{a_l}\big(s, e^{\succ'}\big)\, ds = \nu_{a_l}\big(T_l, e^{\succ'}\big) > q_{a_l} - 1.$$

Then $n_{a_l}(T_l, e^{\succ'}) \geqslant q_{a_l}$ because $n_{a_l}(T_l, e^{\succ'})$ is an integer. It follows that $n_{a_l}(s, e^{\succ'}) \geqslant q_{a_l}$ for all times $s \geqslant T_l$ when a_l is still available under $e^{\succ'}$. Note that a_l is available under $e^{\succ'}$ at $s \geqslant T_l$ if $e_i^{\succ'}(s) = a_l$. Therefore,

$$n_{a_l}\big(s, e^{\succ'}\big) \geqslant q_{a_l} \mathbf{1}_{e_i^{\succ'}(s) = a_l}, \quad \forall s \in [T_l, T_{l+1}).$$

By (6), $\nu_{a_l}(T_l, e^{\succ'}) \geqslant \nu_{a_l}(T_l, e^{\succ}) - \delta(T_l) = q_{a_l} - \delta(T_l)$. Thus

$$\begin{aligned}
\delta(T_l) &\geqslant q_{a_l} - \nu_{a_l}\big(T_l, e^{\succ'}\big) \\
&\geqslant \nu_{a_l}\big(T_{l+1}, e^{\succ'}\big) - \nu_{a_l}\big(T_l, e^{\succ'}\big) \\
&= \int_{T_l}^{T_{l+1}} n_{a_l}\big(s, e^{\succ'}\big)\, ds
\end{aligned}$$

$$\geqslant q_{a_l} \int_{T_l}^{T_{l+1}} \mathbf{1}_{e_i^{\succ'}(s)=a_l}\, ds$$
$$= q_{a_l}\big(\beta(T_{l+1}) - \beta(T_l)\big),$$

where the last equality holds because, by the definition of a_l and T_l, the times in $[T_l, T_{l+1})$ when agent i's consumption in the eating algorithm is $\succ_i$-preferred if the reported preferences change from $\succ$ to $\succ'$ are exactly those when i eats a_l. Thus

$$\beta(T_{l+1}) - \beta(T_l) \leqslant \frac{\delta(T_l)}{q_{a_l}}. \qquad \square$$

Lemma 6. *If $q_a \geqslant M$ for all $a \in \hat{O}$, then*

$$\beta(T_{l+1}) - \beta(T_l) \leqslant \frac{\lambda}{M}\left(1 + \frac{1}{M}\right)^{l-1}, \quad \forall l = 0, 1, \ldots, \bar{l}.$$

Proof. We prove the lemma by induction on l. For $l = 0$, the induction hypothesis holds trivially since $\beta(T_1) = 0$.

Let $l \geqslant 1$. Suppose that the induction hypothesis holds for $0, 1, \ldots, l-1$. We prove that it holds for l.

By the induction hypothesis, if $l \geqslant 2$,

$$\delta(T_l) \leqslant \lambda + \sum_{g=1}^{l-1} \beta(T_{g+1}) - \beta(T_g) \leqslant \lambda + \frac{\lambda}{M}\sum_{g=1}^{l-1}\left(1 + \frac{1}{M}\right)^{g-1} = \lambda\left(1 + \frac{1}{M}\right)^{l-1}. \tag{7}$$

The inequality can be checked separately for $l = 1$.

Since $q_{a_l} \geqslant M$ by assumption, Lemma 5 and (7) imply that

$$\beta(T_{l+1}) - \beta(T_l) \leqslant \frac{\lambda}{M}\left(1 + \frac{1}{M}\right)^{l-1},$$

finishing the proof of the induction step. $\square$

Proof of part (i). Assume that $q_a \geqslant M$ for all $a \in \hat{O}$. If we set $d = \min_{a \succ_i b, a \succeq_i \emptyset} u_i(a) - u_i(b)$ and $D = \max_{a \succeq_i b \succeq_i \emptyset} u_i(a) - u_i(b)$, then

$$u_i\big(PS(\succ)\big) - u_i\big(PS(\succ')\big) = \int_0^1 u_i\big(e_i^{\succ}(s)\big) - u_i\big(e_i^{\succ'}(s)\big)\, ds \geqslant d\gamma(1) - D\beta(1).$$

By definition, $\gamma(1) = \lambda$. Since $\bar{l} \leqslant k$ and $\beta(T_1) = 0$, adding up the inequalities from Lemma 6 for $l = 1, 2, \ldots, \bar{l}$, we obtain

$$\beta(1) \leqslant \sum_{g=0}^{\bar{l}-1} \frac{\lambda}{M}\left(1 + \frac{1}{M}\right)^{g} = \lambda\left(\left(1 + \frac{1}{M}\right)^{\bar{l}} - 1\right) \leqslant \lambda\left(\left(1 + \frac{1}{M}\right)^{k} - 1\right).$$

Therefore,

$$u_i\big(PS(\succ)\big) - u_i\big(PS(\succ')\big) \geqslant \lambda\left(d - D\left(\left(1+\frac{1}{M}\right)^k - 1\right)\right),$$

which is non-negative if

$$M \geqslant \frac{1}{(\frac{d}{D}+1)^{1/k} - 1}. \tag{8}$$

This completes the proof. □

B.2.1. Linearization of the bound (8) as a function of k and D/d

Using Taylor expansions of $(1+x)^{1/k} - 1$ at $x = 0$, we obtain the inequalities[9]

$$\frac{d}{D}\frac{1}{k} - \frac{1}{2}\left(\frac{d}{D}\right)^2 \frac{1}{k}\left(1 - \frac{1}{k}\right) \leqslant \left(\frac{d}{D}+1\right)^{1/k} - 1 \leqslant \frac{d}{D}\frac{1}{k},$$

which lead to a tight bound for the denominator in (8). Hence truth-telling is a weakly dominant strategy for i if

$$M \geqslant \frac{D}{d}\frac{k}{1 - \frac{1}{2}\cdot\frac{d}{D}(1-\frac{1}{k})}.$$

As $d/D \leqslant 1/k$, it follows that

$$k+1 > \frac{k}{1 - \frac{1}{2}\cdot\frac{d}{D}(1-\frac{1}{k})}.$$

Therefore, truth-telling is a weakly dominant strategy for i if

$$M \geqslant (k+1)\frac{D}{d}.$$

B.3. Part (ii)

Define

$$\Lambda = \frac{\lambda}{M}\left(1+\frac{1}{M}\right)^{k-1}.$$

Lemma 7. *Suppose $q_a \geqslant M$ for all $a \in \hat{O}$. Then for all $a \in \hat{O}$ and $t \leqslant T_{\bar{l}}$ with $t + \Lambda \leqslant 1$,*

$$\nu_a\big(t, e^{\succ}\big) = q_a \quad \Rightarrow \quad \nu_a\big(t+\Lambda, e^{\succ'}\big) = q_a.$$

Proof. Assume that $a \in \hat{O}$ and $t \leqslant T_{\bar{l}}$ satisfy $t + \Lambda \leqslant 1$ and $\nu_a(t, e^{\succ}) = q_a$. By inequality (7) in the proof of Lemma 6,

$$\delta(t) \leqslant \delta(T_{\bar{l}}) \leqslant M\Lambda. \tag{9}$$

---

[9] These inequalities can be verified by taking first and second-order derivatives of $(1+x)^{1/k} - 1 - \frac{1}{k}x$ and $(1+x)^{1/k} - 1 - \frac{1}{k}x + \frac{1}{2}\cdot\frac{1}{k}(1-\frac{1}{k})x^2$ for $x \geqslant 0$.

By Lemma 4,

$$\nu_a\big(t, e^{\succ'}\big) \geqslant \nu_a\big(t, e^{\succ}\big) - \delta(t) > q_a - 1. \tag{10}$$

Define $t' = t + \Lambda$. We prove that $\nu_a(t', e^{\succ'}) = q_a$ by contradiction. Assume that $\nu_a(t', e^{\succ'}) < q_a$. Note that $n_a(\cdot, e^{\succ'})$ is increasing on the time interval where a is available under $e^{\succ'}$, hence by (10),

$$n_a\big(t, e^{\succ'}\big) > n_a\big(t, e^{\succ'}\big)t \geqslant \int_0^t n_a\big(s, e^{\succ'}\big)\,ds = \nu_a\big(t, e^{\succ'}\big) > q_a - 1.$$

It must be that $n_a(t, e^{\succ'}) \geqslant q_a$ because $n_a(t, e^{\succ'})$ is an integer. Since a is still available at t' under $e^{\succ'}$, it follows that

$$n_a\big(s, e^{\succ'}\big) \geqslant q_a, \quad \forall s \in [t, t').$$

By (9) and (10),

$$\nu_a\big(t, e^{\succ'}\big) \geqslant \nu_a\big(t, e^{\succ}\big) - \delta(t) \geqslant \nu_a\big(t, e^{\succ}\big) - M\Lambda = q_a - M\Lambda > \nu_a\big(t', e^{\succ'}\big) - M\Lambda.$$

Therefore,

$$M\Lambda > \nu_a\big(t', e^{\succ'}\big) - \nu_a\big(t, e^{\succ'}\big) = \int_t^{t'} n_a\big(s, e^{\succ'}\big)\,ds \geqslant q_a(t' - t) = q_a\Lambda,$$

which contradicts $q_a \geqslant M$. $\square$

Proof of part (ii). Assume that $q_a \geqslant M$ for all $a \in \hat{O}$. The construction of the sequence (a_l, T_l) and the consequence of Lemma 7 that $\nu_{a_{\bar{l}}}(T_{\bar{l}} + \Lambda, e^{\succ'}) = q_{a_{\bar{l}}}$ if $T_{\bar{l}} + \Lambda \leqslant 1$ lead to $u_i(e_i^{\succ'}(s)) \leqslant u_i(e_i^{\succ}(s))$ for all $s > \min\{T_{\bar{l}} + \Lambda, 1\}$.

For technical purposes, we extend the definition of $e_i^{\succ}$ such that $e_i^{\succ}(s) = e_i^{\succ}(0)$ for all $s \in [-\Lambda, 0)$. It follows from Lemma 7 and the observation above that $u_i(e_i^{\succ'}(s)) \leqslant u_i(e_i^{\succ}(s - \Lambda))$ for all $s \in [0, 1]$. We obtain

$$\begin{aligned}
&u_i\big(PS(\succ)\big) - u_i\big(PS(\succ')\big) \\
&\quad = \int_0^1 u_i\big(e_i^{\succ}(s)\big) - u_i\big(e_i^{\succ'}(s)\big)\,ds \\
&\quad = \int_0^1 \max\big\{0, u_i\big(e_i^{\succ}(s)\big) - u_i\big(e_i^{\succ'}(s)\big)\big\}\,ds + \int_0^1 \min\big\{0, u_i\big(e_i^{\succ}(s)\big) - u_i\big(e_i^{\succ'}(s)\big)\big\}\,ds \\
&\quad \geqslant d\lambda + \int_0^1 \min\big\{0, u_i\big(e_i^{\succ}(s)\big) - u_i\big(e_i^{\succ}(s - \Lambda)\big)\big\}\,ds \\
&\quad = d\lambda + \int_0^1 u_i\big(e_i^{\succ}(s)\big) - u_i\big(e_i^{\succ}(s - \Lambda)\big)\,ds
\end{aligned}$$

$$
\begin{aligned}
&= d\lambda + \int_0^1 u_i\big(e_i^{\succ}(s)\big)\,ds - \int_{-\Lambda}^{1-\Lambda} u_i\big(e_i^{\succ}(s)\big)\,ds \\
&= d\lambda + \int_{1-\Lambda}^{1} u_i\big(e_i^{\succ}(s)\big)\,ds - \int_{-\Lambda}^{0} u_i\big(e_i^{\succ}(s)\big)\,ds \\
&= d\lambda - \int_{-\Lambda}^{0} u_i\big(e_i^{\succ}(s)\big) - u_i\big(e_i^{\succ}(s+1)\big)\,ds \\
&\geqslant d\lambda - D\Lambda.
\end{aligned}
$$

Therefore,

$$u_i\big(PS(\succ)\big) - u_i\big(PS(\succ')\big) \geqslant d\lambda - D\Lambda = \frac{d\lambda}{M}\left(M - \frac{D}{d}\left(1 + \frac{1}{M}\right)^{k-1}\right).$$

Suppose that $M \geqslant xD/d$, where $x \approx 1.76322$ solves $x\ln(x) = 1$. Let $\mathbf{e} \approx 2.71828$ denote the base of the natural logarithm. Since $D/d \geqslant k$,

$$\left(1 + \frac{1}{M}\right)^{k-1} < \left(1 + \frac{1}{xk}\right)^{k} = \left(\left(1 + \frac{1}{xk}\right)^{xk}\right)^{1/x} < \mathbf{e}^{1/x}.$$

As $x = \mathbf{e}^{1/x}$, it follows that

$$u_i\big(PS(\succ)\big) - u_i\big(PS(\succ')\big) \geqslant \frac{d\lambda}{M}\left(x\frac{D}{d} - \frac{D}{d}\mathbf{e}^{1/x}\right) = 0.$$

Hence truth-telling is a weakly dominant strategy for i if

$$M \geqslant x\frac{D}{d}. \qquad \square$$

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