EECS 495: Combinatorial Optimization **Lecture 9**
Matroid Intersection

Reading: Schrijver, Chapter 41

Matroid Intersection

Claim: (Edmonds, 1970) For matroids M_1, M_2 on S,

$$\max_{J \in \mathcal{I}_1 \cap \mathcal{I}_2} \{|J|\} = \min_{A \subseteq S} \{r_1(A) + r_2(\overline{A})\}.$$

Proof

Need:

1. deletion
2. contraction
3. submodularity of rank function

Def: The *dual* M^* of matroid $M = (S, \mathcal{I})$ is the matroid with ground set S whose independent sets I are such that $S \setminus I$ contains a basis of M.

Def: The *deletion* $M \setminus Z$ of matroid $M = (S, \mathcal{I})$ and subset $Z \subset S$ is the matroid with ground set $S \setminus Z$ and independent sets $\{I \subseteq S \setminus Z : I \in \mathcal{I}\}$.

Example: Take graph, delete edges, take acyclic subsets of remaining edges.

Def: The *contraction* M/Z is $(M^* \setminus Z)^*$.

Unwrapping, we get:

- M^* is everything that excludes a basis B of M
- $(M^* \setminus Z)$ has ground set $S' = S \setminus Z$ and indep sets is everything that excludes a basis B and excludes Z
- max indep sets of $(M^* \setminus Z)$ is
  - take max indep set of M in Z, say J
  - extend J to basis with elts $J' \in S \setminus Z$
  - then $S \setminus (Z \cup J')$ is max indep set (excludes Z and basis $J \cup J'$)
- so indep sets of $(M^* \setminus Z)^*$ are $\{J' \subseteq S' : J' \cup J \in \mathcal{I}, r(J) = r(Z)\}$

Example: Take graph, contract edges, take acyclic subsets of remaining edges.

Note: Defn does not depend on which indep set $J \subseteq Z$ that we choose (clear from graph example)

Claim: $M' = M/Z$ is a matroid with rank function $r'(A) = r(A \cup Z) - r(Z)$.

Proof: Downward-closed, exchange property both follow as M satisfies them. For rank function:

- let $J' \subseteq A$ be max indep set in $A \subseteq S \setminus Z$ according to M'
- let $J \subset Z$ be max indep set according to M such that $J \cup J'$ indep in M

- then $J \cup J'$ is a max indep set in $A \cup Z$ according to M since if not $\exists e \in A \cup Z, J \cup J' \cup \{e\}$ indep in M
  - if $e \in A$ then J' not max since $J' \cup \{e\} \in \mathcal{I}'$ by defn of contraction
  - if $e \in Z$ then J not max since $J \cup \{e\} \in \mathcal{I}$ by downward closure
- claim follows since $J \cap J' = \emptyset$

Def: A function f is submodular if for any A, B,

$$f(A \cup B) + f(A \cap B) \leq f(A) + r(B)$$

or equivalently if for any $S \subset T$ and $i \notin T$,

$$f(S \cup \{i\}) - f(S) \geq f(T \cup \{i\}) - f(T).$$

Claim: $r(\cdot)$ is rank func of a matroid iff

- $r(\emptyset) = 0$ and $r(A \cup \{e\}) - r(A) \in \{0, 1\}$ for all e, A
- $r(\cdot)$ is submodular

Proof: (of Matroid Intersection Theorem) Already showed max $\leq$ min. Other direction:

$$\max_{J \in \mathcal{I}_1 \cap \mathcal{I}_2} |J| \geq \min_{A \subseteq S} r_1(A) + r_2(S \setminus A)$$

by induction on $|S|$.

- let k be $\min_A(r_1(A) + r_2(S \setminus A)$
- if no $\{e\} \in \mathcal{I}_1 \cap \mathcal{I}_2$ we're done since
  - max ≥ 0
  - for each e, $\{e\} \notin \mathcal{I}_1$ or $\{e\} \notin \mathcal{I}_2$
  - take $A = \{e \in S : r_1(\{e\}) = 0\}$
  - for such A, $r_1(A) + r_2(\overline{A})$ is 0 so min ≤ 0
- else let $\{e\} \in \mathcal{I}_1 \cap \mathcal{I}_2$

  delete e, if min $= k$, (actually just need $\geq$ but min can't grow) we're done since
  - $M_i' = (S' = S \setminus \{e\}, \mathcal{I}_i' = \{J \in \mathcal{I}_i : e \notin J\})$
  - let $A = \operatorname{argmin}_{A \subseteq S'} r_1'(A) + r_2'(S' \setminus A)$
  - by induction, $\max_{J \in \mathcal{I}_1' \cap \mathcal{I}_2'} |J| \geq r_1'(A) + r_2'(S' \setminus A)$
  - common independent sets only grow when add back $\{e\}$ so $\max_{J \in \mathcal{I}_1 \cap \mathcal{I}_2} |J| \geq \max_{J \in \mathcal{I}_1' \cap \mathcal{I}_2'} |J|$
  - by assumption both min equal k

  contract e, if min $\geq k - 1$, we're done since
  - by induction, max $\geq$ min in contracted matroids
  - take common indep set J of size at least $k - 1$ in contracted matroids
  - then $J \cup \{e\} \in \mathcal{I}_1 \cap \mathcal{I}_2$ by defn of contraction and assumption that $\{e\} \in \mathcal{I}_1 \cap \mathcal{I}_2$
  - so max in original matroid $\geq |J \cup \{e\}| \geq k =$ min in original matroid by assumption
- suppose above don't hold. then exist $A, B \subseteq S \setminus \{e\}$ s.t.

  $$r_1(A) + r_2(S' \setminus A) \leq k - 1$$

  and

  $$r_1(B \cup \{e\}) - r_1(\{e\}) + r_2((S' \setminus B) \cup \{e\}) - r_2(\{e\}) \leq k - 2$$

  by submodularity and that $r_1(\{e\}) = r_2(\{e\}) = 1$ we get

  $$r_1(A \cup B \cup \{e\}) + r_1(A \cap B)$$

$$+r_2(S \setminus (A \cap B)) + r_2(S \setminus (A \cup B \cup \{e\}))$$
$$\leq 2k - 1$$

but then either sum of middle two terms or sum of other two terms is at most $k-1$, contradiction that min was k.