EECS 495: Combinatorial Optimization **Lecture 7**
Matroid Representation, Matroid Optimization

Reading: Schrijver, Chapters 39 and 40

Matroids

Recap

Def: A *matroid* $M = (\mathcal{S}, \mathcal{I})$ is a finite ground set $\mathcal{S}$ together with a collection of independent sets $\mathcal{I} \subseteq 2^{\mathcal{S}}$ satisfying:

- downward closed: if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$, and
- exchange property: if $I, J \in \mathcal{I}$ and $|J| > |I|$, then there exists an element $z \in J \setminus I$ s.t. $I \cup \{z\} \in \mathcal{I}$.

Def: A *basis* is a maximal independent set. The cardinality of a basis is the *rank* of the matroid.

Def: *Uniform matroids* U_n^k are given by $|S| = n$, $\mathcal{I} = \{I \subseteq S : |I| \leq k\}$.

Def: *Linear matroids*: Let F be a field, $A \in F^{m \times n}$ an $m \times n$ matrix over F, $S = \{1, \ldots, n\}$ be index set of columns of A. Then $I \subseteq S$ is independent if the corresponding columns are linearly independent.

Note: WLOG any linear matroids can be written as $A = [I_m | B]$ where m is rank of matroid and B is an $(n-m) \times m$ matrix over F.

Def: *Graphic matroids*: Let $G = (V, E)$ be a graph and $S = E$. A set $F \subseteq E$ is independent if it is acyclic.

Food for thought: can two non-isomorphic graphs give isomorphic matroid structure?

Representation

Def: For a field F, a matroid M is *representable* over F if it is isomorphic to a linear matroid with matrix A and linear independence taken over F.

Example: Is uniform matroid U_4^2 binary?

Need: matrix A with entries in $\{0, 1\}$ s.t. no column is the zero vector, no two rows sum to zero over GF(2), any three rows sum to GF(2).

- if so, can assume A is 2×4 with columns 1/2 being $(0, 1)$ and $(1, 0)$ and remaining two vectors with entries in $0, 1$ neither all zero.
- only three such non-zero vectors, so can't have all pairs indep.

Question: representation of U_4^2? $(1, 0), (0, 1), (1, -1), (1, 1)$ in $\Re$.

Def: A *binary* matroid is a matroid representable over $GF(2)$.

Def: A *regular* matroid is representable over any field.

Example: Graphic matroids are regular.

Proof: Take A to be vertex/edge incidence matrix with $+1/-1$ in each column in any order.

- Minimally dependent sets sum to zero perhaps with multiplying by -1.
- Works over any field with $+1$ as multiplicative identity and -1 additive inverse of $+1$.

Note: Have graphic $\subset$ binary $\subset$ regular $\subset$ linear.

Note: There are matroids that are not linear (MacLane, 1936; Lazarson, 1958).

Matroid Operations

Def: (from last lecture): The *dual* M^* of matroid $M = (S, \mathcal{I})$ is the matroid with ground set S whose independent sets I are such that $S \setminus I$ contains a basis of M.

Def: The *deletion* $M \setminus Z$ of matroid $M = (S, \mathcal{I})$ and subset $Z \subset S$ is the matroid with ground set $S \setminus Z$ and independent sets $\{I \subseteq S \setminus Z : I \in \mathcal{I}\}$.

Example: Take graph, delete edges, take acyclic subsets of remaining edges.

Def: The *contraction* M/Z of ... is ... $(M^* \setminus Z)^*$.

[[*So for $X \subseteq Z$ maximal independent set of M, I independent in M/Z if $I \cup X$ independent in M.*]]

Def: If a matroid M' arises from M by a series of deletions and contractions, then M' is a *minor* of M.

Claim: (Tutte, 1958) A matroid is binary if and only if it has no U_4^2 minor.

[[*Similar characterization of ternary matroids as those that exclude the so-called Fano matroid and its dual as a minor.*]]

Conjecture (Rota, 1971): Matroids representable over a finite field can be characterized by a finite list of excluded minors.

[[*Much like planar graphs are those with no $K_{3,3}$ or K_5 as a minor.*]]

Matroid Optimization

Given: Matroid $M = (S, \mathcal{I})$ and weights $c : S \to \mathbb{R}$

Find: max-weight (or min-weight) basis

[[*Recall Kruskal's Alg for min spanning tree: select edges in increasing order of weight*]]

Algorithm: Greedy

- Set $J = \emptyset$.
- Order S s.t. $c_1 \geq \ldots \geq c_n$.
- For $i = 1$ to n, if $J \cup \{i\}$ is independent, $J := J \cup \{i\}$

[[*If weights are non-neg, this is max-weight indep set; otherwise stop selecting elts when c_i becomes negative for max-weight indep set.*]]

Claim: Greedy finds maximal-weight basis.

[[*First rephrase second axiom.*]]

Proof: Clearly a basis. Suppose not max-weight, i.e., for greedy set J and opt J', $c(J) < c(J')$.

- Let $J = \{e_1, \ldots, e_l\}$ be greedy set labeled according to chosen order so $c_{e_1} \geq \ldots \geq c_{e_l}$.
- Let $J' = \{q_1, \ldots, q_k\}$ be max-weight basis labeled s.t. $c_{q_1} \geq \ldots \geq c_{q_k}$.
- Let i be smallest index s.t. $c_{q_i} > c_{e_i}$ (if no such index, must have $k > l$ so let $i = l + 1$).

- Consider independent sets $I = \{e_1, \ldots, e_{i-1}\}$ and $I' = \{q_1, \ldots, q_i\}$.
- since $|I'| > |I|$ exchange property says $\exists z \in I'$ s.t. $I + z$ independent
- but each elt in I' has greater weight than I and z was available to greedy at step i by above, so greedy can't have chosen e_i over z.

[[*In fact, matroids are precisely set systems on which greedy works, see book.*]]

[[*What about running time? Depends on matroid representation to test if $I + z$ independent. Want poly in $|S|$ given indep set oracle, or sometimes given sucinct representation of M like in graphs (note listing all indep sets is exponential in $|S|$). Question, is there a matroid with a sucinct rep in which checking independence is hard?*]]

Matroid Polytopes

Variables: x_s for each $s \in S$ Constraints:

$$x_S \geq 0, \forall s \in S$$

$$\sum_{s \in U} x_s \leq r(U), \forall U \subseteq S$$

Claim: Greedy is optimal.

Claim: Matroid polytope integral.

Proof: Consider primal objective $\max \sum_{s \in S} w(s) x_S$. Dual is:

$$\min \sum_{U \subseteq S} r(U) y_U$$

$$\text{s.t.} \sum_{U : s \in U} y_U \geq w(s), \forall s \in S$$

$$y_U \geq 0, \forall U \subseteq S$$

Let O_P, O_D be primal/dual value. To prove TDI need for any $w \in \mathbb{Z}^n$ exists opt dual soln that's integral.

[[*Recall TDI means for integral cost vector c s.t. primal soln finite, there exists integral opt dual. Furthermore if polytope is TDI and b is integral, then polytope is integral.*]]

- WLOG w non-negative (else discard neg elts and note dual constraint satisfied since $y \geq 0$.
- Let J be independent set found by greedy.
- Note $w(J) \leq \max_{I \in \mathcal{I}} w(I) \leq O_P = O_D$.
- Find integral y s.t. dual value equals $w(J)$ hence proving both claims. Label elts in decreasing order of weight and let $U_i = \{s_1, \ldots, s_i\}$.

$$y_{U_i} = w(s_i) - w(s_{i+1})$$

$$y_{U_n} = w(s_n)$$

$$y_U = 0, \text{ otherwise}$$

  - feasible: for any $s_i \in S$, $\sum_{U : s_i \in U} y_U = \sum_{j=i}^{n} y_{U_j} = \sum_{j=i}^{n-1} (w(s_i) + w(s_{i+1})) + w(s_n) = w(s_i)$.
  - optimal:

$$\begin{aligned}
\sum_{U \subseteq S} r(U) y_U &= \sum_{i=1}^{n-1} r(U_i)(w(s_i) - w(s_{i+1})) \\
&\quad + r(U_n) w(s_n) \\
&= w(s_1) r(U_1) \\
&\quad + \sum_{i=2}^{n} w(s_i)(r(U_i) - r(U_{i-1})) \\
&= w(J)
\end{aligned}$$