EECS 495: Combinatorial Optimization **Lecture 5**
Matching: TDI, Cunningham-Marsh

Reading: Schrijver, Chapter 25

Recap

Primal P:

max $c^T x$ s.t. $Ax \leq b$

Dual D:

min $b^T y$ s.t. $A^T y = c$ and $y \geq 0$

Def: A linear system $\{Ax \leq b\}$ is *totally dual integral* (TDI) if for any integral cost vector for the primal such that max $c^T x, Ax \leq b$ is finite, there exists an integral optimal dual solution.

Theorem 0.1 *(Edmonds-Giles, 1979): If a system $\{Ax \leq b\}$ is TDI and b is integral, then $\{Ax \leq b\}$ is integral (i.e., the extreme points are integral).*

Theorem 0.2 *(Giles-Pullyblank, 1979): For a rational polyhedron $\mathcal{P}$, there exist A and b with A integral such that $\mathcal{P} = \{x : Ax \leq b\}$ and the system is TDI.*

Def: A set of vectors $\{a_i : a_i \in \mathcal{Z}^n\}$ is a *Hilbert basis* if for any integral $c \in cone(a_i) = \{\sum_i \lambda_i a_i : \lambda_i \geq 0\}$, there exist non-negative integers μ_i such that $c = \sum_i \mu_i a_i$.

Theorem 0.3 *The rational system $Ax \leq b$ is TDI iff for each face (actually sufficient to check for each extreme point), tight constraints form a Hilbert basis.*

Theorem 0.4 *Any rational polyhedral cone $C = \{\sum_i \lambda_i a_i : \lambda_i \geq 0, \lambda_i \in \mathcal{R}\}$ with $\{a_i\}$ integral has a finite integral Hilbert basis.*

Note: In fact don't need to assume $\{a_i\}$ integral, follows from rationality of cone.

Integrality of Polytopes

Theorem 0.5 *(Edmonds-Giles, 1979): If a system $\{Ax \leq b\}$ is TDI and b is integral, then $\{Ax \leq b\}$ is integral (i.e., the extreme points are integral).*

Proof: By contradiction.

- Consider extreme point x^* of P s.t. $x^*_j \notin \mathcal{Z}$ for some j.
- Let c be integral vector s.t. x^* unique opt by picking rational vector in cone at x^* and scaling.
- Consider $\hat{c} = c + \frac{1}{q} e_j$ (inside cone for large enough q).
- Since $q\hat{c}^T x^* - qc^T x^* = x^*_j \notin \mathcal{Z}$, either $q\hat{c}^T x^*$ or $qc^T x^*$ not integral.
- By duality and fact that b is integral, one of corresponding dual soln $\hat{y}$ or y not integral.
- Contradicts TDI since both $q\hat{c}$ and qc integral.

Matching Polytope

Def: The *matching polytope* $\mathcal{P}_M$ is the convex hull of incidence vectors $\chi(M) \in \{0,1\}^{|E|}$ of matchings M where $\chi(M)_e = 1$ if $e \in M$ and 0 otherwise.

Def: $\mathcal{P}$ ($\mathcal{P}_2$ from last lecture) is:

- $x_e \geq 0, e \in E$
- $x(\delta(v)) = \sum_{e \in \delta(v)} x_e \leq 1, v \in V$
- $x(E(U)) = \sum_{e \in E(U)} x_e \leq \frac{|U|-1}{2}, U \subseteq V, |U|$ odd

[[*Edmonds gave algorithmic proof of this; we use Cunningham-Marsh, argue that $\mathcal{P}_2$ is TDI.*]]

Primal:

$\max c^T x$ s.t.
$\sum_{e \in \delta(v)} x_e \leq 1, \forall v \in V$
$\sum_{e \in E(U)} x_e \leq \frac{|U|-1}{2}, \forall U \subseteq V, |U|$ odd
$x_e \geq 0, \forall e \in E$

Dual (variables y_v for $v \in V$, z_U for $U \subseteq V$ odd):

$\min \sum_v y_v + \sum_{|U| \text{ odd}} \frac{|U|-1}{2} z_U$ s.t.
$y_u + y_v + \sum_{|U| \text{ odd}, e \in E(U)} z_U \geq c_e, \forall e \in E$
$y, z \geq 0$

[[*TDI says...*]]

Theorem 0.6 *(Cunningham-Marsh, 1978) For all $c \in \mathcal{Z}^{|E|}$, there exists an integral dual solution y, z with value $D(y,z) \leq \nu(c)$ (where $\nu(v)$ is max cost matching).*

[[*Why's this prove TDI, i.e., why are we not implicitly assuming primal value is $\nu(c)$ and hence primal is the matching polytope? I think because duality says primal can't be more than dual...*]]

Proof: By induction on $|V| + |E| + \sum_e c_e$ (recall c integral).

- Assume $c_e \geq 1$ (else delete e) and G connected (else prove for components).
- Base case ($|V| = 2, |E| = 1, c_e \geq 1$): set $y_u = c_e$ and $y_v, z_U = 0$.
- Case 1: $\exists v \in V$ s.t. every max cost matching for c covers v.
  - Modify costs $c'_e = c_e$ for $e \notin \delta(v)$ and $c'_e = c_e - 1$ for $e \in \delta(v)$.
  - Note $\nu(c') = \nu(c) - 1$.
  - By induction, exist integral y', z' feasible for dual with c' s.t. $D(y', z') \leq \nu(c')$.
  - Let $y_v = y'_v + 1$ and $y_u = y'_u$ for $u \neq v$, and $z = z'$.
  - Note y, z feasible since only constraints for $e \in \delta(v)$ changed, and for those both c_e and y_v increased by 1 from c'_e and y'_v.
  - Note further that $D(y,z) = D(y'z') + 1 \leq \nu(c') + 1 = \nu(c)$.
- Case 2: $\forall v, \exists$ max cost matching for c that does not cover v.
- Let $c'_e = c_e - 1$ for all $e \in E$.
- We show all max matchings M for c' miss at least one vertex.
- Let M be max matching for c' with $|M|$ as large as possible.
- Suppose M covers all vertices.
- Let N be max matching for c that does not cover some vertex.
- $c'(N) = c(N) - |N| > c(N) - |M| \geq c(M) - |M| = c'(M) = \nu(c')$ (first inequality because M covers all vertices and N misses at least one)

- Case 2a: Suppose $\exists$ max matching M for c' s.t. $|M| = \frac{|V|-1}{2}$ (i.e., $|V|$ odd and M misses exactly one vertex).
  - By induction, exist integral y', z' s.t. $D(y', z') \leq \nu(c')$.
  - Let $z_V = z'_V + 1$ and $z_U = z'_U$ for all other $U \subset V$; $y = y'$.
  - z_V in every constraint and both z_V and c_e increased by one, so y, z feasible.
  - Also, $D(y, z) = D(y', z') + \frac{|V|-1}{2} \leq \nu(c') + \frac{|V|-1}{2} \leq \nu(c)$ (last inequality follows because can use matching for c' as matching for c)
- Case 2b: All max cost matchings for c' miss at least two vertices.
  - Let M be max cost matching for c' with unmatched vertices u and v s.t. $|M|$ maximized and $d(u, v)$ minimized.
  - Note $d(u, v) \geq 2$ and let t be second node on shortest path from u to v. Note t matched in M (otherwise can add edge (u, t)).
  - Let N be max matching for c, $c(N) = \nu(c)$ such that t unmatched in N.
  - Let P be component of t in $M \Delta N$ and $M' = M \Delta P$ and $N' = N \Delta P$
  - Note M', N' are matchings and $|M'| \leq |M|$ (last edge of path connecting to t is in M).
  - However,
    $c(M) + c(N) = c(M \Delta P) + c(N \Delta P) \rightarrow$
    $c'(M) + |M| + c(N) =$
    $c'(M \Delta P) + |M \Delta P| + c(N \Delta P) \rightarrow$
    $c'(M) + |M| \leq c'(M \Delta P) + |M \Delta P|$
    since $c(N) = \nu(c) \geq c(N \Delta P)$.
  - Since $c'(M) = \nu(c') \geq c'(M \Delta P)$ and $|M| \geq |M \Delta P|$, must be equalities.
  - t unmatched in M'.
  - P can't cover both u and v since neither covered by M and only one can be endpoint of path if covered by N.
  - either u or v unmatched by M', say u
  - then $d(u, t) < d(u, v)$, $|M| = |M'|$, and $c'(M') = c'(M) = \nu(c')$ contradicting our choice of M, u, v.

□

Note: Matching polytope has exponentially many constraints. Has a separation oracle based on minimum odd cut in suitable graph (reading project).

Question: (open): Can one give a compact polyhedral description of the matching polytope, e.g., by suitable lifting of variables? (part of reading project to discuss lifting of variables.)

Matroids

[[Abstracts linear algebra and graph theory.]]
Key set systems to keep in mind:

- subsets of vectors of $\mathcal{R}^n$
- subsets of edges of $G = (V, E)$

Def: A *matroid* $M = (\mathcal{S}, \mathcal{I})$ is a finite ground set $\mathcal{S}$ together with a collection of sets $\mathcal{I} \subseteq 2^{\mathcal{S}}$ satisfying:

- downward closed: if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$, and
- exchange property: if $I, J \in \mathcal{I}$ and $|J| > |I|$, then there exists an element $z \in J \setminus I$ s.t. $I \cup \{z\} \in \mathcal{I}$.

Terminology:

- $I \in \mathcal{I}$ *independent*, $I \notin \mathcal{I}$ *dependent*
- *circuit* is a minimal dependent set of M
- *basis* is a maximal independent set
- I is a *spanning set* if for some basis B, $B \subseteq I$

Example: *Uniform matroids U_n^k*: Given by $|S| = n$, $\mathcal{I} = \{I \subseteq S : |I| \leq k\}$.

Check two properties and see this is a matroid.

What are the...

- bases: sets of size k
- circuits: sets of size $k + 1$
- spanning sets: sets of size at least k

Example: *Linear matroids*: Let F be a field, $A \in F^{m \times n}$ an $m \times n$ matrix over F, $S = \{1, \ldots, n\}$ be index set of columns of A. Then $I \subseteq S$ is independent if the corresponding columns are linearly independent.

Check two properties and see this is a matroid.

What are the...

- bases: minimal sets of vectors that span space spanned by A
- circuits: vectors that span space space spanned by A with one extra
- spanning sets: vectors that span space spanned by A

Example: *Graphic Matroids*: Let $G = (V, E)$ be a graph and $S = E$. A set $F \subseteq E$ is independent if it is acyclic.

Check two properties and see this is a matroid.

What are the...

- bases: minimum spanning trees
- circuits: subgraphs with one cycle
- spanning sets: connected subgraphs that contain every vertex

Note: All bases of a matroid M must have same cardinality.

Def: The *rank function* of M is $r : 2^S \to \mathcal{Z}_+$ given by $r(U) = \max_{I \subseteq U, I \in \mathcal{I}} |I|$.

Note: Corresponds to rank of matrix in linear matroids, hence name.