EECS 495: Combinatorial Optimization **Lecture 3**
Matching: Edmonds-Gallai Decomposition

Reading: Schrijver, Chapter 24

Recap

Def: An *M-flower* is an M-alternating walk that looks like (draw a flower, label stem and blossom, $v_0 \in X$, v_i even, $v_t = v_i$ odd).

Claim: Shortest M-alternating walk is either an M-augmenting path or an M-flower.

Def: Graph G and matching M with M-blossom B give *shrunk graph* G/B with matching M/B (draw flower, shrink blossom).

Claim: M maximum iff M/B maximum in G/B.

Algorithm: Edmonds Matching Algorithm

While $\exists$ alternating walk:

- Find one of minimum length.
- If augmenting, do augmentation.
- Else if a flower with blossom B,
  - Recursively find maximum matching N/B in G/B.
  - If $|N/B| = |M/B|$ then M maximum so return M.
  - Else unshrink N/B to get matching N with $|N| > |M|$ and repeat with $M := N$ and corresponding $X, \hat{G}$.

Edmonds-Gallai Decomposition

Theorem 0.1 *(Tutte-Berge Formula): For any graph G, $\nu(G) = min_{U \subseteq V}(|V| + |U| - o(G-U))/2$.*

[[*Last time, inductive proof. This time, algorithm to find U, an alternate proof.*]]

Def: U is a Tutte-Berge witness if $\nu(G) = (|V| + |U| - o(G-U))/2$.

Def: The *Edmonds-Gallai decomposition* partitions the vertices V of a graph G into sets

- $D(G)$ – set of vertices v such that v is exposed by some maximum matching,
- $A(G)$ – set of neighbors of $D(G)$, and
- $C(G)$ – set of all remaining vertices.

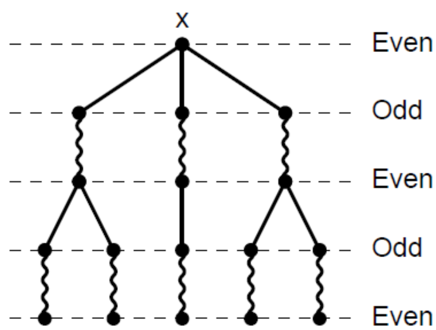
Theorem 0.2 *For the Edmonds-Gallai decomposition,*

- *$U = A(G)$ is a Tutte-Berge witness,*
- *$D(G)$ contains all vertices in odd components in $G-U$, and*
- *$C(G)$ contains all vertices in even components in $G-U$.*

Also, every odd component of $G-U$ is factor-critical (a graph H is factor-critical if for every vertex v there is a matching in H that exposes only v).

Finding decomposition.

Construction: vertices reachable by odd/even alternating paths from a vertex $v \in X$.



Let M be matching returned by Edmonds' Algorithm, X be exposed vertices.

- Even := $\{v : \exists$ even alternating path from X to $v\}$
- Odd := $\{v : \exists$ odd alternating path from X to v and no even one$\}$
- Free := $\{v : \nexists$ alternating path from X to $v\}$

Claim: There is no edge between Even and Free.

Proof: For $u \in$ Even and edge (u, v), v has alternating path from X:

- if $(u, v) \in M$, then even alternating path P to u has (u, v) as final edge, so delete and get odd alternating path to v,
- if $(u, v) \notin M$, then append (u, v) to even alternating path P to u to get odd alternating walk to v, if repeats vertex must be v lies on P so truncate P at v.

so every v adjacent to $u \in$ Even is in Even or Odd.

Let

- G_0 be final graph of Edmonds Algorithm (after shrinking all blossoms iteratively).
- M_0 corresponding max matching found by alg.

Claim: There is no edge within Even in G_0. **Proof:** If $(u, v) \in E$ with $u, v \in$ Even, then

- v has even alternating path P from X since Even,
- v has odd alternating walk P' from X through u (see previous proof).

Append reverse of P to P' to get alternating walk from X to X. Contradiction since M_0 max and G_0 has no flowers(?).

Note: Claim fails in G since all vertices of a blossom are even (we can go around blossom in either direction).

Claim: Even $= D(G) = \{v : \exists$ maximum matching M that exposes $v\}$. **Proof:**

- $(\rightarrow)$: If $v \in$ Even, take M and even path from X to v. Swap edges on path to get maximum matching that exposes v.
- $(\leftarrow)$: If exists M' exposing v, then $M \cup M'$ has even cycles and even paths, and v is endpoint of even path. Since $v \notin M'$, must be $v \in M$ and hence other endpoint of path in X, so $v \in$ Even.

Claim: Odd $= A(G) = \{v : v$ is neighbor of some $u \in D(G)$, but $v \notin D(G)\}$. **Proof:**

- $(\rightarrow)$: If $v \in$ Odd, odd alternating path from X to v and vertex u before v in path is in Even $= D(G)$. Also $v \in$ Odd means $v \notin$ Even by defn.
- $(\leftarrow)$: Everything adjacent to $D(G) =$ Even is in Even$\cup$Odd and $v \notin D(G)$ =Even, so $v \in$ Odd.

Claim: Free $= C(G) = V(G) \setminus (D(G) \cup A(G))$.

Properties

Theorem 0.3 *For the Edmonds-Gallai decomposition,*

- *$U = A(G)$ is a Tutte-Berge witness,*
- *$C(G)$ contains all vertices in even components in $G - U$,*
- *$D(G)$ contains all vertices in odd components in $G - U$, and*
- *every odd component of $G - U$ is factor-critical.*

[[*A graph H is factor-critical if for every vertex v there is a matching in H that exposes only v.*]]

Claim: $C(G)$ is even components.

Proof: We proved no edge between Even and Free, so $C(G)$ disjoint from $D(G)$ in $G - U$. Furthermore, M matches vertices of $C(G)$ to vertices of $C(G)$ so $|M \cap E(C(G))| = |C(G)|/2$:

- $v \in C(G)$ must be matched (else even),
- u matched to v cannot be even since no even-free edges,
- u matched to v cannot be odd since then v would be even,
- hence v is free too.

Claim: $D(G)$ is odd components, each of which is factor-critical.

Proof: For every connected component H of $(G - U) \cap D(G)$, we show:

1. Either $|X \cap H| = 1$ and $|M \cap \delta(H)| = 0$, or $|X \cap H| = 0$ and $|M \cap \delta(H)| = 1$ (where $\delta(H)$ is edges with exactly one endpoint in H).
2. H is factor-critical.

[[*Claim follows – note first condition implies H even-size.*]]

By induction on number of blossoms shrunk.

- Base case: no blossoms shrunk, then $G = G_0$ so we proved
  - no Even-Free edge
  - no Even-Even edge

  so $(G - U) \cap D(G)$ union isolated vertices and claim follows.
- Inductive step: B blossom in G, claim holds for G/B. First condition:
  - in G/B, b even (stem is even alternating path from X to b)
  - in G, all vertices of B even and in same connected component H_b of $(G - U) \cap D(G)$

- since vertices of $B \setminus \{b\}$ matched internally in G, expanding blossom can't change $|X \cap H_b|$ or $|M \cap \delta(H)|$, so first condition holds in G if it holds in G/B

Second condition:

- by induction, all components of $(G - U) \cap D(G)$ other than H_b factor-critical.
- for H_b assume $v \in H_b$ removed.
- if $v \notin B$, by induction is matching M' in G/B that exposes v, so expand it.
- if $v \in B$, by induction is matching M'' in G/B that exposes b, so expand it and take matching in B that exposes v.