EECS 495: Combinatorial Optimization **Lecture 12**
Ellipsoid, Polar Polytopes

Problem: For polytope P,

$$\max\{c^T x | x \in P\}$$

where P may have exponentially many facets.

Goal: Separation iff Optimization

- $\rightarrow$: ellipsoid method
- $\leftarrow$: polar polytopes

Ellipsoid Method

Algs for solving LPs:

- simplex (Dantzig, 40s): practical, not known to be in P
- ellipsoid (Shor; Khachyan, 70s): impracticle, but in P, only requires separation oracle
- interior point (Karmarkar, 80s): practical, in P, require explicit representation of polytope

Idea:

- Take big ellipsoid containing P.
- If center not in P, find separating hyperplane through center dividing ellipsoid in half.
- Consider half-ellipsoid containing P and find new ellipsoid containing this half-ellipsoid.
- Iterate.

Example: Circle at origin, sep hyperplane $x_1 = 0$, draw new ellipsoid (tall, thin).

Fact: Volume of ellipsoids shrinks exponentially.

[[*Hence we are guaranteed to get to center and can bound running time by ratio of initial and final ellipsoid if polytope has positive volume (for other cases, see paper).*]]

Algorithm: Ellipsoid (sketch)

1. Let E_0 be an ellipsoid containing P
2. while center a_k of E_k is not in P do:
   - Let $c^T x \leq c^T a_k$ be s.t. $P \subseteq \{x : c^T x \leq c^T a_k\}$.
   - Let E_{k+1} be min vol ellipsoid containing $E_k \cap \{x : c^T x \leq c^T a_k\}$.
   - $k \leftarrow k + 1$.

Ellipsoids

Recall: A positive definite iff $x^T Ax > 0$ for all non-zero $x \in \mathbb{R}^n$ iff $A = B^T B$ for real matrix B.

Def: Given center a and positive definite matrix A, ellipsoid $E(a, A)$ is $\{x \in \mathbb{R}^n : (x-a)^T A^{-1}(x-a) \leq 1\}$.

Note: Just affine transformations of unit spheres:

- transformation $T(x) = (B^{-1})^T(x-a)$ for $A = B^T B$
- $E(a, A) \rightarrow \{y : y^T y \leq 1\} = E(0, I)$

Shrinking Volume

Claim: $\frac{Vol(E_{k+1})}{Vol(E_k)} < e^{-\frac{1}{2(n+1)}}$

Idea: Show for unit sphere, use transformations (which preserve ratio of volumes).

Claim: For unit sphere E_k and halfspace $x_1 \geq 0$, ellipsoid containing $E_k \cap \{x : x_1 \geq 0\}$ is $E_{k+1} = \{x\}$ s.t.

$$\left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1.$$

Example: In two dimensions, center at $(1/3, 0)$, width $2/3$, height$4/3$.

Proof: For $x \in E_k \cap \{x : x_1 \geq 0\}$,

$$\left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + ...$$

$$= \frac{n^2+2n+1}{n^2} x_1^2 - \left(\frac{n+1}{n}\right)^2 \frac{2x_1}{n+1} + \frac{1}{n^2} + ...$$

$$= \frac{2n+2}{n^2} x_1^2 - \frac{2n+2}{n^2} x_1 + \frac{1}{n^2} + \sum_{i=1}^{n} x_i^2$$

$$= \frac{2n+2}{n^2} x_1(x_1 - 1) + \frac{1}{n^2} + \sum_{i=1}^{n} x_i^2$$

$$\leq \frac{1}{n^2} + \frac{n^2-1}{n^2} \leq 1.$$

Ellipsoid since

- $a = \frac{1}{n+1}(1, 0, \ldots, 0)$
- A = diag matrix with $A_{11} = (\frac{n}{n+1})^2$, $A_{ii} = (\frac{n^2}{n^2-1})$, positive definite (inverse because it's inverse in defn)

Proof: Of vol ratio for these ellipsoids: volume proportional to product of side lengths, so

$$\frac{Vol(E_{k+1})}{Vol(E_k)} = \frac{(\frac{n}{n+1})(\frac{n^2}{n^2-1})^{(n-1)/2}}{1}$$

$$< e^{-\frac{1}{n+1}} e^{\frac{n-1}{2(n^2-1)}} = e^{-\frac{1}{n+1}} e^{\frac{1}{2(n+1)}} = e^{-\frac{1}{2(n+1)}}$$

since $1 + x \leq e^x$ for all x and strict if $x \neq 0$.

Claim: More generally, for unit sphere E_k and halfspace $d^T x \leq 0$ with $||d|| = 1$ (wlog by scaling), ellipsoid $E_{k+1} = E(-\frac{1}{n+1} d, F)$ for $F = \frac{n^2}{n^2-1}(I - \frac{2}{n+1} dd^T)$ contains $E_k \cap \{x : d^T x \leq 0\}$ and ratio of volumes is at most $\exp(-\frac{1}{2(n+1)})$.

Example: For halfspace $x_1 \geq 0$ as above,

- $d = (-1, 0, \ldots, 0)$ so $a = \frac{1}{n+1}(1, 0, \ldots, 0)$ as claimed
- dd^T is matrix with 1 in upper-left, so A_{11} is

$$\frac{n^2}{n^2-1} \left(\frac{n+1}{n+1} - \frac{2}{n+1}\right)$$

$$= \frac{n^2}{(n+1)(n-1)} \left(\frac{n-1}{n+1}\right)$$

$$= \left(\frac{n}{n+1}\right)^2$$

and $A_{ii} = n^2/(n^2-1)$.

Claim: For any E_k and E_{k+1}, ratio of volumes is at most $\exp(-\frac{1}{2(n+1)})$.

Proof:

- Let $E_k = E(a_k, A)$ and $c^T x \leq c^T a_k$ be halfspace containing P.
- Consider transformation $T(x) = (B^{-1})^T(x - a_k)$ where $A = B^T B$.
- Note under T, E_k becomes $E(0, 1)$.
- Note under T, $x = B^T y + a_k$ so halfspace becomes

$$\{y : c^T(a_k + B^T y) \leq c^T a_k\}$$
$$= \{y : c^T B^T y \leq 0\} = \{y : d^T x \leq 0\}$$

  for $d = Bc/\sqrt{c^T B^T Bc} = Bc/\sqrt{c^T Ac}$.
- New ellipsoid in transformed space is $E(-\frac{1}{n+1}d, F)$ for $F = \frac{n^2}{n^2-1}(I - \frac{2}{n+1}dd^T)$.
- Inverse transformation: $E_{k+1} = E(a_k - \frac{1}{n+1}B^T d, B^T F B) = E(a_k - \frac{1}{n+1}b, \frac{n^2}{n^2-1}(A - \frac{2}{n+1}bb^T))$ where $b = B^T d$.

Algorithm: Ellipsoid: For $P = \{x : Cx \leq d\}$,

1. Start with $k = 0, E_0 = E(a_0, A_0)$ where $P \subseteq E_0$.
2. While $a_k \notin P$ do:
   - Let $c^T x \leq d$ be inequality valid for $x \in P$ but $c^T a_k > d$.
   - Let $b = \frac{A_k c}{\sqrt{c^T A_k c}}$.
   - Let $a_{k+1} = a_k - \frac{1}{n+1}b$.
   - Let $A_{k+1} = \frac{n^2}{n^2-1}(A_k - \frac{2}{n+1}bb^T)$.

Analysis: After k iterations, $Vol(E_k) \leq Vol(E_0)\exp(-\frac{k}{2(n+1)})$, so need at most $2(n+1)\ln\frac{Vol(E_0)}{Vol(P)}$ iterations.

Claim: Ellipsoid polytime.

Proof: Show for $S \subseteq \{0, 1\}$ and $P = conv(S)$.

- Assume P full dimensional (else eliminate variables)
- feasibility to optimization:
  - let $c^T x$ be objective func with $c \in \mathbb{Z}^n$ (wlog if c rational).
  - check feasibility of $P' = P \cap \{x : c^T x \leq d + 1/2\}$ and binary search for d in $[-nc_{max}, nc_{max}]$
  - takes $O(\log n + \log c_{max})$ runs of ellipsoid, polynomial
- starting ellipsoid:
  - need to guarantee we contain polytope P, sufficient to contain hypercube
  - for E_0 use ball centered at $(\frac{1}{2}, \ldots, \frac{1}{2})$ of radius $\frac{1}{2}\sqrt{n}$
  - E_0 has volume $(\frac{1}{2}\sqrt{n})^n Vol(B_n)$ where B_n is unit ball and $Vol(B_n) < 2^n$
  - $\log(Vol(E_0)) = O(n \log n)$
- termination: if P' non-empty, not too small (see notes)
- separation oracle (to give halfspace): polytime black-box
- finding optimum soln: get from x' of value at most $d + \frac{1}{2}$ to x of value exactly d by finding any extreme point x with $c^T x \leq c^T x'$ (see notes)

Applying Ellipsoid

Problem: Maximum weight matching

Recall: Matching polytope

$$\sum_{e \in E(S)} x_e \leq \frac{|S| - 1}{2}, \forall |S| \text{ odd}$$

$$\sum_{e \in \delta(v)} x_e \leq 1, x_e \geq 0$$

Goal: Separation oracle.

- Given x^*, last two constraints easily checked.
- Others checked with sequence of min cuts.

Claim: There's a polytime separation oracle.

Proof: Assume $|V|$ even (else add a vertex).

- Let $s_v = 1 - \sum_{e \in \delta(v)} x_e$ (slack of constraint for v).
- Note $\sum_{e \in E(S)} x_e \leq (|S| - 1)/2$ becomes
$$\sum_{v \in S} s_v + \sum_{e \in \delta(S)} x_e \geq 1.$$
- Let $H = (V \cup \{u\}, E \cup \{(u, v) : v \in V\})$ new graph with new vertex u connected everywhere.
- Let capacity u_e of edge be x_e if $e \in E$ or s_v for $e = (u, v)$.
- Note $\sum_{v \in S} s_v + \sum_{e \in \delta(S)} x_e \geq 1$ iff $\sum_{e \in \delta_H(S)} u_e \geq 1$.
- Thus just need to find min cut in H among cuts $S \subseteq V$ with $|S|$ odd; if value is ≥ 1, x^* feasible, else found violation.
- This is the min T-odd cut problem and is polytime.

Polar Duality

Def: Given polytope $C \subseteq \mathbb{R}^n$ containing origin, find representation s.t.

$$C = \{c | a_i \cdot c \leq b_i\},$$

where $b_i = 1$ (scale constraints). The *polar* of C is $C^* = conv(a_1, \ldots, a_k)$

Example:

1. C is unit circle, polar is unit circle.
2. C is square with corners $(1, 1), (1, -1), (-1, 1), (-1, -1)$.
   Polar is diamond with corners $(1, 0), (0, 1), (-1, 0), (0, -1)$.
3. C is bulging rectangle with corners $(100, 3), (100, -3), (-100, 3), (-100, -3)$.
   Polar is tall thin rectangle with corners at $(+/-1/100, 0), (0, +/-1/3)$.

Note: Facets become vertices and vice versa. Size/shape reverses.

Claim: Polars have following properties:

- $(C^*)^* = C$.
- If C is origin-symmetric, so is C^*.
- If $A \subseteq B$ then $B^* \subseteq A^*$.
- If A is scaled up, A^* is scaled down.

Def: If $C \subseteq \mathbb{R}^n$, the *polar* of C is the set $C^* = \{x \in \mathbb{R}^n : x^T c \leq 1 \forall c \in C\}$.

Claim: Two defns are equiv.

Proof: Exercise.

Claim: $(C^*)^* = C$.

Proof: $C \subseteq C^{**}$:

- $C^* = \{x : x^T c \leq 1 \forall c \in C\}$ and $C^{**} = \{y : y^T x \leq 1 \forall x \in C^*\}$.
- Let y be point in C.
- By defn of polar of C, for all $x \in C^*$, $x^T y \leq 1$.

- By defn of polar of C^*, conclude $y \in C^{**}$.

$C^{**} \subseteq C$:

- Assume not and let $y \in C^{**}$ be s.t. $y \notin C$.
- Since $y \in C^{**}$ have $y^T x \leq 1$ for all $x \in C^*$.
- Since $y \notin C$ there's separating hyperplane v with $x^T v \leq 1$ for $x \in C$ and $y^T v > 1$.
- By first condition, $v \in C^*$ and so second contradicts $y \in C^{**}$.

So to separate over polar, optimize over polytope.