EECS 495: Combinatorial Optimization **Lecture 1**
Matching: König's Theorem, Tutte-Berge Formula

Reading: Schrijver, Chapters 16 and 24

Logistics

- Website: linked to from my homepage
- Lectures: M/W 1.30-3, some F 1.30-3 [[*Reschedule for 10.30-noon?*
- Readings: from Schrijver, Combinatorial Optimization: Polyhedra and Efficiency
- Problem Sets: list of problems on website; work through some fraction by end of term
- Other Work: propose and solve a problem that would be appropriate for a problem set, do a reading project plus half-hour presentation
- Content: matchings, matroids, submodularity. Based on course by Michel Goemans.

Matching

Definitions

Def: A *matching* is a set of edges that share no vertices.

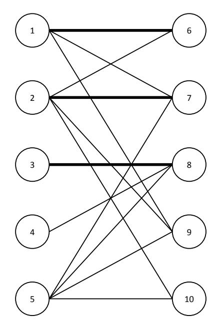
Def: A vertex v is *covered* by a matching if v is incident with an edge in the matching.

Def: A vertex is *exposed* by a matching if it is not covered.

Def: A matching that covers every vertex is a *perfect matching* or a 1-*factor*. [[*A d-factor of a graph is a d-regular spanning subgraph.*]]

Def: A *vertex cover* is a set of vertices C such that every edge is incident with at least one vertex in C.

Example:



In figure,

- Matching: $\{(1,6),(2,7),(3,8)\}$
- Covered: $\{1,2,3,6,7,8\}$
- Exposed: $\{4,5,9,10\}$
- Vertex cover: $\{1,2,3,4,5\}$

König's Theorem

Claim: If M is a matching and C is a vertex cover, then $|M| \leq |C|$. **Proof:** Counting argument:

- at least one endpoint of each edge in M must be in C since C covers all edges,
- edges don't share vertices since M matching, so $|M| \leq |C|$.

Def: The cardinality of the maximum matching is $\nu(G) = \max_M |M|$.

Def: The cardinality of the minimum vertex cover is $\tau(G) = \min_C |C|$.

Claim: For any graph G, $\nu(G) \leq \tau(G)$. **Proof:** By above claim.

Theorem 0.1 *(König's Theorem): For bipartite graphs G, $\nu(G) = \tau(G)$.*

Example: Matching $\{(1,6),(2,7),(3,8),(5,10)\}$ is of maximum size since there is a vertex cover $\{1,2,5,8\}$ of the same cardinality.

Proof: Constructive.

Def: An *alternating path* with respect to M is a path that alternates between edges in M and $E - M$.

Def: An *augmenting path* with respect to M is an alternating path in which the first and last vertices are exposed.

Example: Paths $4-8-3$, $6-1-7-2$, and $5-7-2-6-1-9$ are alternating, but only last one is also augmenting.

Note: If there's an augmenting path P that contains k edges of M, then it contians $k+1$ edges not in M. Hence can increase cardinality of M by setting $M' = (M-P) \cup (P-M)$.

Claim: M is maximum if and only if there are no augmenting paths with respect to M (true for non-bipartite as well). [[*Proof is an exercise.*]]

Algorithm: Finding a maximum matching:

- Start with empty matching.
- Repeatedly augment current matching along augmenting path if one exists.

Let A denote lhs, B denote rhs.

Algorithm: Finding an augmenting path:

- Direct an edge from A to B if not in M.
- Direct an edge from B to A if in M.
- Create vertex s and edges from s to each exposed vertex in A.
- Do BFS from s to find exposed vertex in B.

Example: Finds augmenting path $5-7-2-6-1-9$.

Analysis:

- runtime: at most $\nu(G) \leq n/2$ iterations, each iteration at most m steps, so $O(nm)$.
- correctness: there's an augmenting path iff there's a directed path between exposed vertex in A and exposed vertex in B.

Proof: (of König's Theorem). Run alg. When terminates,

- let L be set of vertices reachable from exposed vertex in A (e.g., this is $\{3,4,8\}$ in example), and

- let C be $(A - L) \cup (B \cap L)$ (e.g., this is $\{1, 2, 5, 8\}$ in example).

Then C is a vertex cover and $|C| = |M|$ where M is matching returned by alg.

- C is a vertex cover: consider an edge (a, b).
  - if $a \notin L$, then covered by $A - L$.
  - if $a \in L$ and (a, b) not an edge in matching $b \in L$ too so covered by $B \cap L$.
  - if $a \in L$ and (a, b) an edge in matching, then (a, b) only incoming edge to a so must have $b \in L$ and hence covered by $B \cap L$.
- $|C| = |M|$: we show $|C| \leq |M|$ since reverse is always true.
  - no vertex in $A - L$ is exposed by definition of L,
  - no vertex in $B \cap L$ is exposed since otherwise we'd have an augmenting path so alg wouldn't have terminated, and
  - no matching edge between $a \in A - L$ and $b \in B \cap L$ since otherwise a would be in L.

  Hence every vertex in C is matched by a distinct edge in M.

Tutte-Berge Formula

Example: Non-bipartite graphs may have $\nu(G) < \tau(G)$, e.g., 3-cycle.

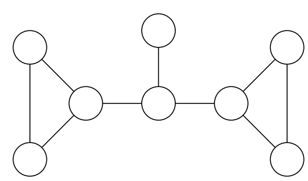
For graph G and $U \subseteq V$,

- let $G - U$ be subgraph obtained by deleting vertices in U, and
- $o(G - U)$ be number of components of G that contain an *odd* number of vertices.

Idea: Consider a matching M in $G - U$:

- It leaves at least one vertex of each odd component unmatched.
- In G these perhaps can be matched to vertices in U, but this can happen at most $|U|$ times.
- Hence matchings in G leave at least $o(G - U) - |U|$ vertices exposed.

Example: Let U be middle vertex in figure.



Theorem 0.2 *(Tutte-Berge Formula): For any graph G, $\nu(G) = min_{U \subseteq V}(|V| + |U| - o(G - U))/2$.*

Proof: Suppose G connected (formula's additive). Do induction on number of vertices.

Base case: one vertex, trivial.

Case 1: G contains vertex v covered by *all* maximum matchings (e.g., middle vertex in example).

- Then $\nu(G - \{v\}) = \nu(G) - 1$.
- By induction, Tutte-Berge Formula holds in $G - \{v\}$ for some set U'.
- Let $U = U' \cup \{v\}$. Then

$$\begin{aligned}
\nu(G) &= \nu(G-v)+1 \\
&= (|V-v| + |U-v| - o(G-v-(U- \\
&= (|V|-1+|U|-1-o(G-U))/2+ \\
&= (|V|+|U|-o(G-U))/2.
\end{aligned}$$

Case 2: for every vertex v there is a maximum matching M that does not cover v (e.g., 3-cycles).

Claim: Each maximum matching leaves exactly one vertex exposed.

Hence $\nu(G) = (|V|-1)/2$ and Tutte-Berge Formula follows by choosing $U = \emptyset$.

Proof: (of claim): By contradiction: suppose each maximum matching leaves two vertices exposed.

Choose maximum matching M and two exposed vertices u and v such that distance $d(u,v) \geq 2$ is minimized over all choices of (M, u, v).

[[*Distance is at least 2 since if it's 1 we can add an edge contradicting maximality of M.*]]

Let t be intermediate vertex on shortest $u-v$ path and N a maximum matching that exposes it whose symmetric difference with M is minimal.

By minimality of (M, u, v), N must cover u and v, so there is some other vertex x that it does not cover which is covered by M.

Let y be vertex matched to x by M and note $y \neq t$ (otherwise could add to N).

Let z be vertex matched to y by N and note $z \neq x$ (since x unmatched by N).

Then $N - yz + xy$ is a matching that exposes t and has smaller symmetric difference with M contradicting choice of N.

□

Example: (for proof)