EECS 495: Combinatorial Optimization
Instructor: Nicole Immorlica

This is a list of exercises related to material discussed during lectures. The list will be updated throughout the quarter. Students are expected to complete 15 of the exercises by the end of the quarter.

1. Use König's theorem to prove that a k-regular bipartite graph has a perfect matching.

2. Let M be a matching of G and let $\nu(G)$ be the cardinality of a maximum matching. Show that there are at least $\nu(G) - |M|$ vertex-disjoint M-augmenting paths. Use this result to conclude that if M is a matching of cardinality 4000 in a graph having a matching of size 5000, then there is an M-augmenting path of length at most 9.

3. Prove Petersen's Theorem, namely, if every node of G has degree 3 and $G \setminus e$ is connected for every edge e, then G has a perfect matching. Hint: First show that if $U \subseteq V$, $|U|$ odd, then $|\delta(U)| \geq 3$.

4. An *edge cover* of a graph $G = (V, E)$ (having no isolated vertices) is a subset F of edges such that every vertex is incident to at least one edge in F. Show how a minimum cardinality edge cover can be determined from a maximum matching. Hence prove that the minimum cardinality of an edge cover is $|V| - \nu(G)$.

5. *Slither* is a two-person game played on a graph $G = (V, E)$. The players, called *First* and *Second*, play alternately, with First playing first. At each step, the player whose turn it is chooses a previously unchosen edge. The only rule is that at every step the set of chosen edges forms a path. The loser is the first player unable to make a legal play at his or her turn. Prove that if G has a perfect matching, then First can force a win. Extend this result showing that if the set $C(G) = \emptyset$ in the Edmonds-Gallai decomposition, then First can force a win.

6. Let $Ax \leq b$ be a system of linear inequalities with A and b rational. Show that there exists a positive integer t such that $(1/t)Ax \leq (1/t)b$ is TDI.

7. Derive the Tutte-Berge formula from Cunningham-Marsh.

8. Let $G = (V, E)$ be a graph. Prove that the convex hull of characteristic vectors of forests of G is the set of all x satisfying:

$$\sum_{e=(u,v):u,v \in T} x_e \leq |T| - 1, \forall T, \emptyset \subset T \subseteq V,$$

$$x_e \geq 0, \forall e \in E.$$

9. Let $G = (V, E)$ be a graph whose edges E are the ground set S of the following set systems. Determine which systems are matroids. If the system is a matroid, give a proof. Else give a counter-example.

   (a) $\mathcal{I} = \{J \subseteq E :$ each component of the subgraph (V, J) contains at most one circuit (i.e., simple cycle)$\}$

   (b) $\mathcal{I} = \{J \subseteq E :$ each component of the subgraph (V, J) contains at most one circuit and no even circuit$\}$

   (c) $\mathcal{I} = \{J \subseteq E :$ each component of the subgraph (V, J) contains at most one circuit and no odd circuit$\}$

10. Prove that for a matroid $M = (S, \mathcal{I})$ with cost vector $c \in \mathbb{R}^{|S|}$, a subset $J \in \mathcal{I}$ is a maximum-weight independent set with respect to c if and only if

    (a) $e \in J$ implies $c_e \geq 0$;

    (b) $e \notin J$ and $J \cup \{e\} \in \mathcal{I}$ implies $c_e \leq 0$;

    (c) $e \notin J, f \in J, J + e - f \in \mathcal{I}$ implies $c_e \leq c_f$.

11. Given a full rank matrix $A \in \mathbb{R}^{n \times n}$ (true also for any field F), let R and C denote the indices of the rows and columns of A. Given $I \subset R$, show using matroid intersection that there exists $J \subset C$ with $|I| = |J|$ such that both $A(I, J)$ and $A(R \setminus I, C \setminus J)$ are of full rank.

12. Consider the following graph orientation problem: we would like to orient the edges of a graph G in such a way that each vertex has at most k incoming edges. Prove that this is possible if and only if $|E[W]| \leq k|W|$ for each subset of vertices W.

13. Prove that in the algorithm for matroid intersection presented in Lecture 10,

    (a) $r_2(S \setminus U) = |I \setminus U|$,

    (b) and $I \Delta P \in \mathcal{I}_2$.

    (The similar statements with the first matroid required for the correctness of the algorithm were proved in class.)

14. Prove that the greedy algorithm presented in Lecture 11 yields a feasible optimal solution.

15. Let f be a submodular function such that $f(\emptyset) = 0$ and let $A_1, \ldots, A_t \subseteq A$ be a collection of sets such that each element of A appears in exactly k of these sets. Prove that $\sum_{i=1}^{k} f(A_i) \geq k f(A)$.

16. Prove Matroid Base Covering. Specifically, let $M = (S, \mathcal{I})$ be a matroid with rank function r. For $k \in \mathbb{N}$, S can be covered by k independent sets if and only if $k \cdot r(U) \geq |U|$ for all $U \subseteq S$.

17. Show that for a graph $G = (V, E)$, the edges E can be partitioned into k forests if and only if for all $U \subseteq V$, $|E[U]| \leq k(|U| - 1)$.

18. Show that a graph $G = (V, E)$ contains k edge-disjoint spanning trees if and only if for every partition P of V into n sets $V_1, \ldots, V_n$, the number of edges crossing the partition P is at least $k(n - 1)$.

19. Prove that a Shannon switching game over M is a neutral game if and only if M contains two disjoint co-spanning trees and the distinguished element e is a member of one of them.

20. Consider a modified version of the special Shannon switching game (recall the special game is when the join player seeks to establish a basis in the matroid) where for every one move that the join player makes, the cut player is allowed to cut k edges during their turn.

    (a) Prove that if a matroid M is a join game in this setting, then there must be at least k bases of the matroid.

    (b) Show by counter-example that the converse is false. That is, find a matroid that has k bases that is not a join game in this setting.

21. For the Shannon switching game, find a characterization of join games in the following cases:

    (a) Join and Cut both choose k edges per turn.

    (b) Join chooses j edges per turn; Cut chooses k edges per turn.

22. Show that if M is a matroid representable over a field F, then any minor of M is representable over F. Hint: it's easy to see that a deletion of M is representable over F; it only remains to show that the dual M^* is too.

23. Consider the following randomized algorithm for the Combinatorial Auction problem. Given submodular valuation functions v, find the solution $x^*(v)$ to the following relaxation of the problem (note that the objective function is not linear):

$$\begin{array}{lll} \text{maximize}_x & v(x) = \sum_i \sum_S v(S) \prod_{j \in S} x_j \prod_{j \notin S} (1 - x_j) & \\ \text{subject to} & \sum_i x_{ij} \leq 1 \quad \forall j & (1) \\ & 0 \leq x_{ij} \leq 1 \quad \forall i, j. & (2) \end{array}$$

    Then use *Poisson Rounding* to round $x^*(v)$ to a distribution over integral solutions. The purpose of this problem is to analyze whether such an approach defines a polynomial-time truthful mechanism with a good approximation for social welfare.

(a) Is the above objective function concave for submodular functions (note that this implies the algorithm is polynomial time since we know that convex programs are polynomial time solvable)?

(b) Is the above algorithm MIDR?

Hint: To prove the algorithm is MIDR, you should prove that $\forall v, v', E_{x \sim \text{pr}(x^*(v))}[v(x)] \geq E_{x \sim \text{pr}(x^*(v'))}[v(x)]$, where $pr(x)$ is the Poisson Rounding distribution on x.

(c) Does the above rounding give a $1 - 1/e$-approximation for submodular functions?

Hint: To prove the algorithm is a $1 - 1/e$-approximation, you should prove that $v(x)(1 - 1/e) \leq E_{x' \sim \text{pr}(x)}[v(x')]$, for all x that satisfies (1) and (2). Note that the algorithm discussed in the class needed this to be true only for integral points that satisfy (1) and (2), so we only proved that in the class. Now we should check to see if it holds for all points satisfying (1) and (2).

Discussion (not necessary for solving the problem): The above is a natural approach that we use to solve problems: Relax the problem, find the optimal relaxed solution, and then round it to (distributions of) integral solutions. If the answers to all 3 questions are *Yes*, then the above algorithm can be converted to a $(1 - 1/e)$-approximation truthful mechanism for Combinatorial Auctions problem. But we know that the paper discussed in the class did not use this approach, and instead used the the novel approach of *maximizing over rounded outcomes*. This is justified if the traditional approach does not work. So if you prove that the answer to any of the above 3 is *No*, then you have justified the paper.