Representability of matroids

Greg McGlynn

March 9, 2011

1 Overview

There are several interesting problems in the field of matroid representation theory. Nice results include the characterization of binary, ternary, and quaternary matroids by excluded minors. It remains an open question whether you can characterize the matroids representable over a given finite field by a list of excluded minors.

Here we give the proof that the binary matroids are exactly those that exclude the minor U_4^2 and also an example of a matroid that has no representation over any field.

2 Binary matroids

We start with a review of some stuff from class.

Definition 1. *A representation of a matroid M is a matrix A with entries from some field F such that there is a one-to-one correspondence between the columns of A and the ground set of M, and a set of columns in A is linearly independent (as vectors) iff the corresponding set is independent in M.*

Definition 2. *A matroid is binary if it is has a representation over the field GF(2).*

Definition 3. U_4^2 *is the matroid with a ground set of cardinality 4 in which every 2-element set is a basis.*

Theorem 1. U_4^2 *is not binary*

Proof. In the past we have argued that a matroid of rank n only requires n rows in a representation. U_4^2 has rank 2. But there are only 3 distinct non-zero 2-element vectors over GF(2)–not enough to represent the 4 elements of the ground set. □

Definition 4. *A minor of a matroid M is a smaller matroid obtained by a sequence of deletions and contractions in M.*

It is not too hard to show that if M has a representation over a field F, then so does any minor of M.

Theorem 2. *A matroid M is binary if and only if it does not have* U_4^2 *as a minor.*

Clearly if M is binary, then it cannot have U_4^2 as a minor, because that would imply that U_4^2 is binary also. It remains to show that if M is not binary, then it contains a U_4^2 minor. The strategy is:

- Construct a binary matroid N with a specific relationship to the non-binary M.
- Show that this relationship implies that either M or N contains a U_4^4 minor.
- Conclude that M contains a U_4^2 minor.

Here is the proof.

Proof. Suppose M is a non-binary matroid with ground set S. We construct a related binary matroid N, represented over GF(2) by a set of column vectors $\{x_s \mid s \in S\}$.

- First pick a basis B of M. Make B a basis of N also by choosing $\{x_b \mid b \in B\}$ to be a set of linearly independent vectors over GF(2).
- For each $s \in S \backslash B$ there is a unique $C_s \subset B + s$ that is a circuit in M. Define $x_s = \sum_{b \in C_s - s} x_b$. By construction, C_s is a cicruit in N as well.

Now $\{x_s\}$ defines a binary matroid N.

Claim: There is no $Y \subset S$ such that $|B \triangle Y| = 2$ and Y is a basis of exactly one of M, N.

Proof of claim: If $|B \triangle Y| = 2$ then $Y = B - b + s$ for some $b \in B$, $s \in S \backslash B$. In both M and N, $B + s$ contains the same unique circuit C_s. Then $B + s - b$ is a basis in M or N iff $b \in C_s$ so that removing b breaks the circuit. So Y cannot be a basis of only one of M, N.

This is the relationship between the non-binary M and the binary N outlined in the strategy. Now proving the following lemma will complete the theorem. □

Definition 5. *Suppose M and N are matroids over a common ground set S. Call $X \subset S$* **wrong** *if it is a basis of only one of M, N. Call $B \subset S$ a* **far basis** *if it is a common basis of both M and N and there is no $Y \subset S$ such that Y is wrong and $|B \triangle Y| = 2$.*

Lemma 1. *Suppose M, N are distinct matroids over a common ground set S that have a far basis B. Then at least one of M, N contains a U_4^2 minor.*

Proof. By contradiction. Suppose there is a far basis B, yet neither M nor N has a U_4^2 minor.

Choose B a far basis, and X wrong, to minimize $|B \triangle X|$ (which must be ≥ 4).

- Now if $S \backslash (B \cup X) \neq \emptyset$ then we can delete $S \backslash (B \cup X)$. In the deletions, B is still a far basis, so $M \backslash (S \backslash (B \cup X))$ and $N \backslash (S \backslash (B \cup X))$ are counterexample matroids.
- Similarly, if $B \cap X \neq \emptyset$ then we can contract $B \cap X$ and get even smaller counterexamples.
- By repeating this procedure we can get some smaller counterexamples M, N (minors of the original matrices) with $B \cap X = \emptyset$ and $B \cup X = S$.

Without loss of generality suppose X is a basis of M only. Note that X is now the only wrong set of S. If there were a different wrong set Y, then $|B \triangle Y| < |B \triangle X|$, violating the assumed minimality of $|B \triangle X|$.

We now find a basis D of M with $|B \triangle D| = 2$. Pick an $x \in X$. $B + x$ has a unique circuit in M. Remove one of the elements of the circuit that is a member of B, say b. Now $D = B + x - b$ is a basis of M because it contains no circuit. $D \neq X$ (since $|B \triangle D| = 2$, while $|B \triangle X| \geq 4$). Therefore D is common basis of M and N.

Note that $|D \triangle X| < |B \triangle X|$. We chose $|B \triangle X|$ to be minimum such that

- B is a far basis and
- X is wrong.

D and X threaten to break this minimum. It must be that D is not a far basis. Thus there is some wrong Y with $|D \triangle Y| = 2$. Since X is the only wrong thing around, we conclude that $|D \triangle X| = 2$.

Now from $|B \cap X| = 0$, $|B \triangle D| = 2$, $|D \triangle X| = 2$, we conclude that $|S| = |B \triangle X| = 4$ and $|B| = |X| = 2$.

Concretely, let $S = \{a, b, c, d\}, B = \{a, b\}, X = \{c, d\}$. We have assumed that $M \neq U_4^2$, so there is some 2-element subset of S that is dependent in M. Without loss of generality suppose it is $\{a, c\}$.

Now some elementary reasoning that leads to a contradiction:

- By DC on B, $\{a\}$ is independent in M.
- By EX on $\{a\}$ and X, $\{a, d\}$ is independent in M.
- By DC on X, $\{c\}$ is independent in M.
- By EX on $\{c\}$ and B, $\{b, c\}$ is independent in M.
- Both $\{a, d\}$ and $\{b, c\}$ must be independent in N also, or else they would wrong and have a symmetric difference 2 with B, a far basis.
- By DC on $\{b, c\}$, $\{c\}$ is independent in N.
- Use the exchange property on $\{c\}$ and $\{a, d\}$ in N. Either: $\{a, c\}$ is independent in N, but not in M, which contradicts the definition of B as a far basis since they have symmetric difference 2; or $\{c, d\} = X$ is independent in N, contradicting the definition of X as wrong.

So we reach a contradiction from assuming that $M \neq U_4^2$ – that is, from the assumption that neither of the original matroids contained a U_4^2 minor. So one of them must have. This completes the proof that the binary matroids are exactly those that have no U_4^2 minor. □

3 A non-representable matroid

Next we look at representations of some specific matroids and construct a matroid that is not representable over any field. The figures represent two matroids: the non-Fano matroid and the Fano matroid. In the figures, the vertices represent the elements of the ground set. The bases of the matroids are all 3-element sets except those with a line through them. A simple check of the exchange property verifies that these rules and the figures define two matroids.

Below is a representation of the Fano matroid F_7 over GF(2). Interestingly, it is also a representation of the non-Fano matroid F_7^-

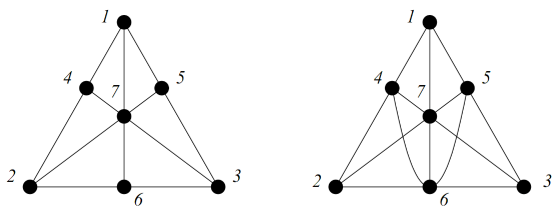

Figure 1: On the left, the non-Fano matroid. On the right, the Fano matroid. (taken from [3])

if the field is taken to be R.

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

In fact, we can show that if $M \in \{F_7, F_7^-\}$ is representable over some field F, then the above matrix is a representation over F. Assuming M is representable over our field F, let's build a representation of M. $B = \{1, 2, 3\}$ is a basis, so we can make the first 3 columns the orthogonal unit vectors, as in A. Next we look at the unique circuits contained in $B + s$, $s \in \{4, 5, 6, 7\}$. For example, $\{1, 2, 4\}$ is a minimal dependent set, so the 4th column must have non-zero entries in the first and second row, and a zero entry in the third. Carrying on in this way we find that our representation must have non-zero entries distributed as below:

$$\begin{pmatrix} 1 & 0 & 0 & * & * & 0 & * \\ 0 & 1 & 0 & * & 0 & * & * \\ 0 & 0 & 1 & 0 & * & * & * \end{pmatrix}$$

We can scale rows and columns to set most of these entries to 1. Scale the columns to set the first nonzero entry in each column equal to 1. Next scale rows 2 and 3 to set the last column equal to all 1s. Finally scale columsn 2, 3, and 6 again to set their first nonzero entries equal to 1. We are left with only 3 undetermined entries:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & a & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & b & c & 1 \end{pmatrix}$$

Finally the values of a, b, and c are constrained to all be 1 by the fact that $\{3, 4, 7\}$, $\{2, 5, 7\}$, and $\{1, 6, 7\}$ are all dependent. So if there is any representation, then our original matrix A is one.

Interestingly, the above reasoning never relied on whether $\{4, 5, 6\}$ was independent or dependent. So the above applies equally well to F_7 and F_7^-. What this tells us is that there is no field over which there is a representation of both F_7 and F_7^-. If there were, both matroids would be represented by A on that field. But then they would have to be the same matroid. Since they are different matroids, there can be no such field.

This makes it easy to create a matroid with no representation over any field. Define the direct sum of matroids $M = (S_1, I_1)$ and $N = (S_2, I_2)$ defined over disjoint ground sets S_1, S_2 by

$$M \oplus N = (S_1 \cup S_2, \{X \cup Y \mid X \in I_1, Y \in I_2\}).$$

Then M and N are both minors of $M \oplus N$, for instance $M = (M \oplus N) \backslash N$. Therefore if $M \oplus N$ is representable over some field F, so are M and N. From this it follows that $F_7 \oplus F_7^-$ cannot be represented over any field.

4 Problems

1. Show that if M is a matroid representable over a field F, then any minor of M is representable over F (hint: it's easy to see that a deletion of M is representable over F; it only remains to show that the dual $M^\star$ is too. A contraction is just the operation dual + deletion + dual).

5 References

- (1) Michel Goemans's lecture notes at http://math.mit.edu/ goemans/18438/lec9.pdf
- (2) A. Schrijver. Combinatorial Optimization: Polyhedra and Efficiency, volume B. Springer, 2003.
- (3) J. Oxley. What is a matroid? Cubo, 5:179218, 2003.
- (4) J.G. Oxley. Matroid Theory. Oxford University Press, 1992.