Shannon Switching Game

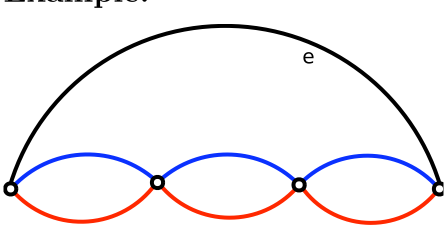
In the Shannon switching game, two players, *Join* and *Cut*, alternate choosing edges on a graph G. Join's objective is to create a path from some vertex to another; Cut's objective is preventing him from accomplishing this.

The game is easily extended to playing over matroids rather than just paths through graphs. Given a matroid $M = (S' = S + e, \mathcal{I})$, each player chooses respectively elements from the ground set S rather than the edge set. e is an unplayable edge, join wins if he can select a set of edges to form a cycle including e, cut wins if he can prevent Join from doing that.

Definition 0.1 *In Shannon's switching game over a matroid $M = (S' = S + e, \mathcal{I})$, Join wins if he can choose a set $S_J \in S$ s.t. $S_J + e \subseteq \mathcal{C}(M)$, otherwise Cut wins.*

Definition 0.2 *We call a game a* Join game *if the Join player always wins, a* Cut game *if Cut always wins and a* neutral game *if whomever goes first wins.*

Example:



Consider the given graph. Join can always win, even if he goes second, just by always playing the edge opposite of the edge that cut is playing. If you remove one edge though, whomever goes first wins by playing that edge. Remove two red (or blue) edges and it becomes a cut game - try it!

In order to build up to the general case, let us consider (and prove things) about a special case of the game, where each player is trying to construct a basis of the entire graph, and the first player to do this wins.

Definition 0.3 *In the Shannon special switching game over a matroid $M = (S, \mathcal{I})$, Join wins if he chooses a set $S_J \in S$ s.t. $S \in span(S_J)$, otherwise Cut wins.*

Definition 0.4 *Base packing: A matroid M contains 2 bases if and only if $\forall\ T \subset S$: $|S \setminus T| \geq 2(r(S) - r(T))$.*

Definition 0.5 *Strong Exchange property: B_1, B_2 are bases of M. Then for all $x \in B_1$, there exists a y such that $B_1 - x + y$ and $B_2 - y + x$ are still bases in M.*

Theorem 0.1 *A matroid $M = (S, \mathcal{I})$ is a join game iff M contains 2 disjoint bases.*

Proof: If M is a join game, then it contains 2 disjoint bases.
We shall assume that this isn't true and then show that cut has a winning strategy if he

plays first. If M does not contain 2 bases, then by the Base Packing theorem, there is a set T such that $|S \setminus T| < 2(r(S) - r(T))$. If cut plays elements from this set, join will not be able to create a basis and thus cut can win. This is a contradicts that M is a join game.

If M contains 2 disjoint bases, then M is a join game.
We will show that the join player can focus on playing basis B_1. If at any point, the cut player chooses an edge from B_1, then the join player can choose the corresponding y in B_2 that makes $B_1 - x + y$ a basis and continue to establish a basis. After any given round where we remove 1 edge (what cut plays) and contract another (what join) plays, $B_1 - x$ and $B_2 - y$ are bases in the new matroid, so we can repeat this strategy until join plays the whole basis and wins the game.

Duality

Recall the definition of a Dual Matroid:

Definition 0.6 *If $M = (S, \mathcal{I})$ is a matroid, the dual $M^* = (S, \mathcal{I}^*)$ is a matroid, where $\mathcal{I}^* = \{I \subseteq S | S \setminus I$ is a spanning set of $M\}$*

- The dual of a planar graph forms a matroid, with spanning trees.
- The complement of a base in M is called a *co-base* in M^*

Lemma 0.2 *The rank function $r_{M^*}()$ satisfies $r_{M^*}(U) = |U| + r_M(S \setminus U) - r_M(S)$*

Proof in lecture 6 notes.

Claim: The game is symmetric between the players over the duals - whenever the cut player is playing, he plays as if he is the join player playing on the dual.

Using this, we provide a more precise definition of winning the game:

Definition 0.7 *The join player wins if he can construct a dependent set including e first in the primal, the cut player wins if he can construct a dependent set including e in the dual.*

We'll prove this by showing first that if we have a cycle in the primal and the dual, their intersection cannot have only one edge in it. Since

Lemma 0.3 $C \in \mathcal{C}(M), D \in \mathcal{C}(M*) \Rightarrow |D \cap C| \neq 1$

This gives us the intuition for playing in the dual - if they choose sets and keep playing until one or both of them form a cycle, the set of only one will span e.

Lemma 0.4 *Exactly one player wins.*

That both can't win is implied by previous lemma. If join doesn't win, we can use the dual rank property to show that cut must win. Assume join doesn't win, so $e \notin \text{span}(S_J) \to r(S_J + e) = r(S_J) + 1$. Most of proof is due to Lemma 0.2.

Thus if Join doesn't win, Cut must win.

Corollary 0.5 *A game is a* Cut game *iff $\exists A, B \subseteq S$ such that $A \cap B = \emptyset$, $e \in span_{M*(A)} = span_{M*(B)}$.*

We'll now give a complete characterization of short, cut and neutral games:

Theorem 0.6 *Let $M = (S' = S + e, \mathcal{I})$ be the graph of a Shannon Switching game in which e is the unplayable edge. Then exactly one of the following conditions holds:*

1. *M contains two disjoint cospanning trees spanning but not containing e. (join game)*
2. *M^* contains two disjoint cospanning trees spanning but not containing e. (cut game)*
3. *M contains two disjoint cospanning trees and e is a member of one of the trees. (neutral game)*

Theorem 0.7 *A matroid M is a join game iff there exists disjoint A,B with $e \in span(A) = span(B)$*

Overview of proof

- The left direction is simple and relies on the same strategy as the previous proof. We simply discard elements from $A \cup B$ until A and B become bases and the result follows.
- Right direction: First define X which minimizes $2r(U) - |U|$
- If $e \in span(X)$, then we show there must exist these two disjoint sets A and B with e in the span and we are done.
- If $e \notin span(X)$ then we show that cut has a winning strategy, which contradicts assumption.

Lemma 0.8 *Conditions of base packing theorem are equivalent to $2r(S') - |S'| = \min_{U \subseteq S'} 2r(U) - |U|$*

Proof:

- $|S \setminus U| \geq 2(r(S) - r(U))$ by base packing
- $|S| - |U| \geq 2(r(S) - r(U))$ because $U \subseteq S$
- $2r(U) - |U| \geq 2r(S) - |S|$ for all $U \subseteq S$

Lemma 0.9 *Given matroid $= (S', \mathcal{I})$ and set $F \subseteq S'$ there exists two disjoint subsets A,B with $F = span(A) = span(B)$ if F is a minimizer of $2r(U) - |U|$ where $U \subseteq F$.*

Proof: An extension of the base packing theorem with the restriction operator limiting the matroid to elements of F. F being a minimizer gives us 2 bases that satisfy the needed properties.

Let X be the largest set that minimizes $\min_{U \subseteq S} 2r(U) - |U|$

Theorem 0.10 *$e \in span(X)$ implies $\exists A, B \subset S, A \cap B = \emptyset, e \in span(A) = span(B)$*

Proof: By 0.9, X being a minimizer gives the exists of disjoint A and B such that span(A) = span(B) = X. Following from the definition of span(), we have that span(A) = X implies span(span(A)) = span(X) = span(A). So if $e \in span(X)$ it is also in span(A) and span(B)

X is not necessarily a global minimizer because it was chosen from S not $S \cup e$. If we assume $e \notin span(X)$ then X is a global minimizer.

We will need the following lemma about the set of the minimizers to complete our proof

Lemma 0.11 *The set of minimizers to $2r(U) - |U|$ is a lattice.*

Proof: Observe that $2r(U) - |U|$ is submodular because $r(U)$ is submodular and $|U|$ is modular. The result follows from applying submodular inequalities.
This gives that any set that minimizes $2r(U) - |U|$ will be contained in X because we obtain X from repeated unions.

Lemma 0.12 *$e \notin span(X)$ implies X is a global minimizer of $2r(U) - |U|$*

Proof: By contradiction. We assume there exists a Y such that $2r(X) - |X| > 2r(Y) - |Y|$. Then Y must contain e. Then $Y - e \subseteq X$, so $2r(Y - e) - |Y - e| \geq 2r(X) - |X|$. We combine these two inequalities to show $r(Y - e) = r(Y)$, yielding $e \in span(Y - e)$. This contradicts that $e \notin span(X)$.

Definition 0.8 *Rank in Dual Matroid:* $r_{M*}(U) = |U| + r(S \setminus U) - r(S)$

Lemma 0.13 *If U is a minimizer to $2r_{M*}(U) - |U|$ for $U \subseteq S$ then $S \setminus U$ is a minimizer to $2r(Y) - |Y|$ for $Y \subseteq S$*

Proof: Follows from application of above definition.

Lemma 0.14 *If U is a minimizer to $2r_(U) -|U|$ for $U \subseteq S$ then $S \setminus U$ is a minimizer to $2r_{M*}(Y) - |Y|$ for $Y \subseteq S$*

Theorem 0.15 *If M is a join game, then there exists disjoint A and B with $e \in span(A) = span(B)$*

Proof:

- Assume not. By 0.10 this is equivalent to $e \notin span(X)$
- If $e \notin span(X)$ by 0.12 X is a global minimizer to $2r(U) - |U|$
- Then $S \setminus X$ minimizes $2_{M*}r(U) - |U|$
- Then by 0.9 there exists two disjoint subsets A', B', such that $S \setminus X = span(A') = span(B')$
- Since $e \notin X$, $e \in S \setminus X = span(A') = span(B')$
- We can now use a similar strategy to the original theorem to show that Cut can construct an S_C that $e \in span_{M*}(S_C)$
- By previous claim this implies that cut can win in the dual game if he moves first, which implies that join cannot win. This contradicts that this game is a join game. Thus these sets must exist.

Problems

Problem: Prove the third part of Theorem 0.6, that a game over M is a neutral game iff M contains two disjoint cospanning trees and e is a member of one of them.

Problem:

(a) Consider a modified version of the Special Shannon switching game (recall the special game is when the join player seeks to a establish a basis in the Matroid) where for every one move that the join player makes, the cut player is allowed to cut k edges during their turn. Prove that if a matroid M is a join game in this setting, then there must be at least k bases of the matroid.

(b) Show by counter-example that the converse is false. Find a matroid that has k bases that is not a join game in this modified version of the game.

Problem: Find a characterization of Join games in one or more of the following cases:

(a). Join and Cut both choose k edges per turn.

(b). Join choses j edges per turn, Cut chooses k edges per turn.