MATROID UNION

ALECK JOHNSEN

EECS 495: Combinatorial Optimization
Reading Project Presentation
February 28th, 2011
Professor N. Immorlica
Northwestern University

1. Partition Matroids

Given set system $M = (E, \mathcal{I})$.

- Let $E = E_1 \uplus \ldots \uplus E_n$, each with paramater $k_i \in \mathbb{N}_0$
- Let $\mathcal{I} := \{X \subseteq E : |X \cap E_i| \leq k_i \forall i\}$
- Then M is a 'partition' matroid

Proof:

- DC Holds: If we remove elements from some $X \subseteq \mathcal{I}$, no $|X \cap E_i|$ will increase
- Exchange Holds: If $I, J \in \mathcal{I}$, $|I| < |J|$, then $\exists E_i$ s.t. $|J \cap E_i| > |I \cap E_i|$. I can add any element of $(J \setminus I) \cap E_i$ and still be indep.

Notes:

- Because we can set $k_i = 0$ for some E_i, we can equivalently not require the E_i's to cover E
- M would not be a matroid if the E_i's were not disjoint
- Example?

2. Matroid Union

Def: The <u>union</u> of matroids $M_1 \vee \ldots \vee M_k$ is $(S_1 \cup \ldots \cup S_k, \mathcal{I}_1 \vee \ldots \vee \mathcal{I}_k)$, where $\mathcal{I}_1 \vee \ldots \vee \mathcal{I}_k := \{I_1 \cup \ldots \cup I_k : I_i \in \mathcal{I}_i\}$

2.1. Examples.

It's natural to think of matroid unions as combinations on disjoint S_i's:

- The super ice cream sunday: choose 3 of 31 flavors and 4 of 18 toppings
- The possible choices are equivalent to the indep. sets of $M = U^3_{31} \vee U^4_{18}$
- This is also a partition matroid

The combinations can also be on overlapping but not identical ground sets:

- Professors A and B share a lab and can hire up to three students each
- Students have submittted applications to A or to B or to both
- Feasible hiring choices are equivalent to the indep. sets of $M = U^3_a \vee U^3_b$
- $S_a \cap S_b$ is not empty so this is NOT a partition matroid

The ground sets can also be identical, and $M \vee M$ is often interesting:

- Consider M^k, the union of k copies of a graphic matroid M with itself
- The largest indep. set in M^k has cardinality $|E|$ *iff* E can be covered by exactly k forests
- M^k is no longer a graphic matroid!

What are the independent sets of M^k for M a matching matroid?

2.2. Comparison to Matroid $\cap$.

- $\cap$ requires ground sets of M_1, M_2 to be identical
- $\cup$ places no requirements on the S_i's
- $\cap$ computes a new $\mathcal{I}$ by $\cap$ on $\mathcal{I}_1, \mathcal{I}_2$
- $\cup$ computes a new $\mathcal{I}$ by 'multiplying' the elements of $\mathcal{I}_1, \mathcal{I}_2$
- the result of matroid $\cap$ is probably not a matroid
- the result of matroid $\cup$ **is** a matroid, and our next goal is to prove this

2.3. **Lemma 1.** Let $M' = (S', \mathcal{I}')$ be any matroid, with rank function r'. For any $f : S' \to S$, define

$$\mathcal{I} = \left\{ f(I') : I' \in \mathcal{I}' \right\} \tag{1}$$

(with $f(I') = \{f(s) : s \in I'\}$). Then $M = (S, \mathcal{I})$ is a matroid with rank

$$r(U) = \min_{T \subseteq U} \left(|U \setminus T| + r'(f^{-1}(T)) \right) \tag{2}$$

for all $U \subseteq S$.

Proof: (*M is a matroid*)

DC:

- Consider $I \in \mathcal{I}$.
- Then $\exists$ some $I' \in \mathcal{I}'$ s.t. $f(I') = I$
- Also, any subset of I is indep. in M because it can be expressed as $f(I'')$ for some $I'' \subseteq I'$

Exchange:

- Consider $I, J \in \mathcal{I}, |I| < |J|$
- Choose $I', J' \in \mathcal{I}'$ s.t. $f(I') = I, f(J') = J, |I| = |I'|, |J| = |J'|$, and $|I' \cap J'|$ is maximized
- M' a matroid $\Rightarrow \exists j \in J' \setminus I'$ s.t. $I' + j \in J'$
- $f(j) \notin f(I')$, else $\exists$i$\in I'$ s.t. $f(j) = f(i)$, but replacing I' with $I' - i + j$ would increase $|I' \cap J'|$
- So $f(j) \in J \setminus I$ with $f(I' + j) \in \mathcal{I}$

Proof: (*of the rank function*)

Fix a $U \subseteq S$, and write $f(U) = U'$.

Construct a partition matroid on the original ground set S':

- Set $P = (S', \mathcal{I}_P)$ induced by partition sets $(f^{-1}(e) : e \in U)$ and all $k_i = 1$
- $\mathcal{I}_P = \left\{ I' \subseteq U' : |f^{-1}(e) \cap I'| \leq 1 \forall e \in U \right\}$
- $r_P(T') = |\{e \in U : f^{-1}(e) \cap T' \neq \varnothing\}| =$ # of partitions covered by $T' \subseteq U'$

Now $I \subseteq U$ has $I \in \mathcal{I}$ *iff* $\exists I' \subseteq U'$ s.t.

- $f(I') = I$
- $I' \in \mathcal{I}'$
- $I' \in \mathcal{I}_P$

Why is the last statement true?

It is true, therefore

$$r(U) = \max_{I \subseteq U, I \in \mathcal{I}} |I| = \max_{I' \subseteq U', I' \in \mathcal{I}' \cap \mathcal{I}_P} |I'| \tag{3}$$

From matroid interesection we get

$$r(U) = \min_{T' \subseteq U'} \left(r'(T') + r_P(U' \setminus T') \right) \tag{4}$$

$$= \min_{T \subseteq U} \left(r'(f^{-1}(T)) + |U \setminus T| \right) \tag{5}$$

which is our goal. The last equality holds because:

- Consider Eq. (4) and any $T' \subseteq U'$
- What is r_P?
- We are looking for a min
- For each partition set E'_i, if $|T' \cap E'_i| < |U' \cap E'_i|$, we should remove $T' \cap E'_i$ from T'
- This will not change $r_P(U' \setminus T')$ and can not increase $r'(T')$
- If we do this for all E'_i's, we transform T' into T'' s.t. T'' is of the form $f^{-1}(T)$ for some $T \subseteq U$
- Since Eq. (4) achieves a min on some T'', we can instead search over $T \subseteq U$ and we get Eq. (5)

2.4. **Theorem: Matroid Union.** Let $M_1 = (S_1, \mathcal{I}_1), ..., M_k = (S_k, \mathcal{I}_k)$ be matroids with respective rank functions $r_1, ...r_k$. Then $M = M_1 \vee ... \vee M_k$ is a matroid with rank function

$$r(U) = \min_{T \subseteq U} \left(|U \setminus T| + \sum_{i=1}^{k} r_i(T \cap S_i) \right) \tag{6}$$

Proof:

We force disjoint S_i's: for each M_i we create $M'_i = (S'_i, \mathcal{I}'_i)$ by relabeling $e \in S_i$ as (i, e). $M''_i = M'_1 \vee ... \vee M'_k$ is a matroid

- DC: An $I'' \in \mathcal{I}''$ is a union $I'_1 \uplus ... \uplus I'_k$ with $I'_i \in \mathcal{I}'_i$. Clearly a subset of I'' can be written as the union of subsets of the I'_i's.

- Exchange: $\forall I'', J'' \in \mathcal{I}'', |I''| < |J''|$: S_i''s disjoint $\Rightarrow$ $\exists i$ s.t. $|J'' \cap S_i'| > |I'' \cap S_i'|$. Also, these are both indep. sets in $I_i' \Rightarrow$ $\exists j \in (J'' \cap S_i')$ s.t. $(I'' \cap S_i') + j \in I_i' \Rightarrow$ $I'' + j \in \mathcal{I}''$

Note: to determine the rank of a subset U'' in M'', we would intersect U'' with each of the disjoint S_i''s and apply the respective r_i's, and return the sum.
Let $f : S'' \to S$ be a selector of $e, f(i, e) = e$.
Use Lemma 1 to transform M'' to M, then M is a matroid by M'' a matroid.
Also by Lemma 1:

(7)
$$r_M(U) = \min_{T \subseteq U} \left(r_{M''}(f^{-1}(T)) + |U \setminus T| \right)$$

(8)
$$= \min_{T \subseteq U} \left(\sum_{i=1}^{k} r_{M''} \left(f^{-1}(T) \cap S_i' \right) + |U \setminus T| \right)$$

(9)
$$= \min_{T \subseteq U} \left(\sum_{i=1}^{k} r_i(T \cap S_i) + |U \setminus T| \right)$$

which is the desired result. Note that the first equality holds by our comment before on $r_{M''}$ and the second equality holds by $r_{M''}(f^{-1}(T) \cap S_i') = r_i(T \cap S_i)$:

- Consider $T \cap S_i$
- S_i' constructed one-for-one from S_i
- Under f^{-1}, each $t \in T \cap S_i$ maps to a value in S_i' and possibly other values.
- By necessity, all 'excess' values reached by $f^{-1}(T)$ are not in S_i', therefore $f^{-1}(T \cap S_i)$ does map one-to-one bijectively if its image is restricted to $f^{-1}(T) \cap S_i'$

3. Corollaries and Applications

These results will be used by Greg and Darrell for Shannon's switching game.

3.1. **Corollary 1.** Let $M = (S, \mathcal{I})$ be a matroid with rank function r. For $k \in \mathbb{N}$, the max size of the union of k indep. sets is

$$r_{M^k}(S) = \min_{U \subseteq S} (|S \setminus U| + k \cdot r_M(U)) \quad (10)$$

Proof:
The result follows trivially from the Matroid Union Theorem on $M_1 = ... = M_k = M$, with the S_i's and r_i's then all identical.

- Example: $M = U_6^2$.
- Then $M^k = M \vee M \vee M = U_6^6$.
- What subset achieves a min for $r_{U_6^6}$ in Eq. (10)?
- Example: M is a matching matroid on a 'star' graph.
- What subset of vertices achieves a min in Eq. (10)?

3.2. **Corollary: Matroid Base Packing.** Let $M = (S, \mathcal{I})$ be a matroid with rank function r. For $k \in \mathbb{N}$, there exist k disjoint bases of M *iff*

$$k \cdot (r(S) - r(U)) \leq |S \setminus U| \quad (11)$$

$\forall U \subseteq S$
Proof: M has k disjoint bases *iff* the max size of a union of k indep. sets equals $k \cdot r(S)$. By Corollary 1, this is true *iff*

$$\min_{U \subseteq S} (|S \setminus U| + k \cdot r(U)) = k \cdot r(S) \quad (12)$$

Fix any U. Then we can replace the `min` function and 'equality' in (12) with '$\forall U \subseteq S$' and '$\geq$' respectively, and rearranging gives us the result.

- Example: Let $\sigma(M)$ be the largest possible number of disjoint bases
- Let $\lambda(M)$ be the smallest minimal subset of E whose deletion reduces the rank of M
- Then $\sigma(M) \leq \lambda(M)$ for any matroid
- Example: Common base packing
- Can two matroids on the same S be partitioned into 'common' bases?

3.3. Corollary: Basis Exchange. Let $M = (S, \mathcal{I})$ be a matroid. For any two bases of M, B_1 and B_2, and for any partition of B_1 into sets $X_1, Y_1, \exists$ a partition of B_2 into X_2, Y_2 s.t. $X_1 \cup Y_2$ and $Y_1 \cup X_2$ are bases of M.
(Note the similarity to Strong Basis Exchange, which is a special case)
Proof:
Notes on matroid contraction:

- $M/Z = (M^* \setminus Z)^*$
- $r_{M/Z}(X) = r(X \cup Z) - r(Z)$ for $X \subseteq S \setminus Z$
- Contraction on a matroid yields a matroid
- Useful way to use it:
  For $Z \subseteq S, X$ a base of Z, we have:
  $I \subseteq S \setminus Z$ has $I \in \mathcal{I}_{M/Z}$ *iff* $I \cup X \in \mathcal{I}_M$

Let $M_1 = M/Y_1, M_2 = M/X_1$
Then $X_1 \in \mathcal{I}_1$ and $Y_1 \in \mathcal{I}_2$
Why? Plug into formula above:

- $X = Z = Y_1, I = X_1$. Given these:
- Y_1 indep. in $M \Rightarrow X$ a base of Z
- $I \subseteq S \setminus Z$ by $I \uplus Z = B_1 \in \mathcal{I}_M$, which then directly implies $X_1 = I \in \mathcal{I}_{M/Y_1}$

By symmetry for $Y_1 \in \mathcal{I}_2$
Therefore B_1 is indep. in

$$\begin{aligned} M_1 \vee M_2 = &(S, \mathcal{I}_1 \vee \mathcal{I}_2) \\ &(I_1 = X_1, I_2 = Y_1) \end{aligned}$$

Now we only need to show $B_2 \in \mathcal{I}_1 \vee \mathcal{I}_2$, since this gives us a way to properly partition B_2. Why is this sufficient?
Using the Matroid Union Theorem, we compute $r_{M_1 \vee M_2}(B_2) =$
(13)

$$\min_{U \subseteq B_2} (|B_2 \setminus U| + r_{M_1}(U \setminus Y_1) + r_{M_2}(U \setminus X_1))$$

Using the contraction rank formula above, we replace r_{M_1} and r_{M_2} to get $r_{M_1 \vee M_2}(B_2) =$
(14)

$$\min_{U \subseteq B_2} (|B_2 \setminus U| + r_M(U \cup Y_1) - r_M(Y_1) + r_M(U \cup X_1) - r_M(X_1))$$

Here, first note that $X_1, Y_1 \in \mathcal{I}_M \Rightarrow$ that their ranks equals their cardinalities, resp. Then using submodularity on $r_M(U \cup Y_1)$ and $r_M(U \cup X_1)$, we get $r_{M_1 \vee M_2}(B_2) \geq$
(15)

$$\min_{U \subseteq B_2} (|B_2 \setminus U| + r_M(U \cup Y_1 \cup X_1) + r_M(U) - |Y_1| + |X_1|)$$

But $r_M(U \cup Y_1 \cup X_1) = r_M(U \cup B_1) = |B_1| = |Y_1| + |X_1|$.
Also, by $U \subseteq B_2, U \in \mathcal{I}_M$ and $r_M(U) = |U|$, so the final reduction is $r_{M_1 \vee M_2}(B_2) \geq |B_2|$
With $r_U \leq |U|$ for all rank functions, we get equality, which in turn implies $B_2 \in \mathcal{I}_1 \vee \mathcal{I}_2$, and we are DONE.

4. Exercises

We give problems so that you can test your understanding.

4.1. Prove Matroid Base Covering. Specifically, let $M = (S, \mathcal{I})$ be a matroid with rank function r. For $k \in \mathbb{N}$, S can be covered by k independent sets *iff* $k \cdot r(U) \geq |U| \forall U \subseteq S$.

4.2. Show that for a graph $G = (V, E)$, the edges E can be partitioned into k forests *iff* $\forall U \subseteq V, |E[U]| \leq k(|U| - 1)$.

4.3. Show that a graph $G = (V, E)$ contains k edge-disjoint spanning trees *iff* for every partition P of V into n sets $V_1, ..., V_n$, the number of edges crossing the partition P is at least $k(n-1)$.

(Hint: for Exercises 2 and 3, one of them can be derived from Base Packing, and the other from Base Covering.)